Modifying the ABCs of Number Theory

A Preprint

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April 18, 2022

Contents

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ABSTRACT

The abc-conjecture (Masser and Oesterlé) is an active problem that has remained open for several decades. The conjecture would imply a direct proof of Fermat's Last Theorem and also shine new light on the mystery of prime gaps. The statement of the conjecture concerns the asymptotic distribution of triples of integers (a, b, c), which satisfy: (1) $a + b = c$, and (2) a, b, c are divisible by few primes for their size (they are "round"). Such triples are measured using a quality metric; triples with highest quality are considered special. The standard quality, which is difficult to analyze, implicitly employs the concept of the Geometric Mean. I created several new classes of quality metrics that utilize the Doubly Geometric Mean (DGM) instead of the Geometric Mean. Through detailed investigation, I demonstrated that these metrics have qualitatively similar behavior to the standard quality, while privileging round triples. Furthermore, I connected Mersenne primes to high-quality triples. I also developed efficient algorithms to calculate large, high-quality triples, and implemented these algorithms in Python. My analysis allowed me to determine, with high efficiency, numerous triples with high qualities and millions of digits within a fraction of a second. These triples contribute new insights toward proving the abc-conjecture, which has applications in and outside of theoretical mathematics. I also formulated several conjectures about the new quality metrics, and illustrated the connection between these conjectures and various unsolved problems in mathematics, such as the twin prime conjecture and the Lenstra–Pomerance–Wagstaff conjecture on Mersenne primes.

1 Introduction

The abc-conjecture, originally introduced by Joseph Oesterlé [10] and David Masser [11] in the 1980s, has remained one of the most active, difficult, and popular conjectures in mathematics for decades. The conjecture concerns groups of integers (a, b, c) with $a + b = c$, where all of a, b, c have few prime divisors. Such groups of integers are called abc-triples. The conjecture has connections to several other conjectures, involving elliptic curves, the distribution of primes, the Fermat-Catalan conjecture and the Beal conjecture. There have been several recent works in proving theoretical results that provide insight towards the abc-conjecture [9] and developing computer algorithms to find high-quality triples quickly [4, 5, 6]. Nitaj [7] summarizes several of these results. Several algorithms have been developed to compute high-quality triples; van der Horst [8] provides an excellent introduction to such algorithms and introduces an algorithm using elliptic curves. Finally, Elkies [2] illustrates connections between the conjecture and several other unsolved problems in number theory.

In Section 2, we provide background on the abc-conjecture, Fermat's Last Theorem, and motivation for looking for high-quality, round triples. We define the "standard" quality metric, q_s. This quality is essential to the typical formulation of the abc-conjecture, which we state explicitly in this section. We state several known results concerning this standard quality, and observe the asymptotic behavior of this quality. We then tabulate a few well-known high quality values, and define terminology related to high-quality triples. These observations will contribute to the analysis that we conduct in the later sections and serve as important foundations for new quality metrics. We will expand on these foundations in the following sections.

Section 3 defines a new quality metric using the Doubly Geometric Mean. This quality metric is different from the standard quality metric as it privileges triples that use extremely few primes in their factorizations. The standard quality also accomplishes this task, but this new Doubly Geometric Mean quality pushes this idea even further. Furthermore, this quality metric is easier to analyze asymptoptically than the standard quality. We analyze this quality in a considerable level of detail, conducting several tests and computing quality values for a very large number of triples with size up to 24,000,000 digits, using just a standard laptop. We investigate the asymptotic behavior of the quality, and develop three methods to compute high-quality triples, using brute force, primes powers, and Mersenne primes. The third method is, computationally, the most effective, and we use it to compute a triple with quality greater than 4000.

Sections 4 and 5 provide extensions on the Doubly Geometric Mean Quality from Section 3. We define several new classes of quality metrics that use various parameters to control their growth and the roundness of the resulting high-quality triples. We investigate the behavior of the quality metrics when we change these parameters, and produce general results about the metrics in these classes. We use these classes to refine our search for high-quality, round triples.

Section 6 describes the algorithms used to calculate high-quality triples using the metrics defined in the previous sections. We develop three different algorithms to find high-quality triples, and compare their time complexities. We arrive at an extremely efficient approach, the Mersenne Prime Algorithm, and use this algorithm to calculate large, high-quality triples for the DGM Quality in section 3. Section 7 provides insight on two other quality metrics that we defined, which involve the Divisor Function and the Harmonic Mean. These metrics have quite different behaviors from the metrics in sections 3, 4, and 5.

We conclude by explaining the significance of this research towards the abc-conjecture itself, and explain the connection between this research and several unsolved problems and conjectures in Number Theory.

The main contributions of this work are as follows:

1. We created several new quality metrics (Definitions 3.2, 7.1, 7.4) that are in the same spirit of the standard quality for the abc-conjecture, which is a 40-year-old, open problem. These new quality metrics quantify the roundness of an abc-triple (a, b, c).
2. We extended our new quality metric to include several classes of metrics (Definitions 4.1, 5.1) that account for important characteristics of an abc-triple, such as its smoothness. Each element of these classes is a quality metric, and every such metric is unique.
3. We analyzed all the quality metrics that we defined in detail, and constructed several families of triples with reasonably high quality. We proved several theorems about the growth rate of the quality metrics evaluated on these families (Theorems 3.8, 3.9, 3.10, 3.12, 3.14).
4. We were able to show an asymptotic improvement on the brute force algorithm (Algorithm 1) to calculate high-quality triples using the Mersenne prime algorithm (Algorithm 3). We implemented our algorithms using Python, and were able to use the Mersenne prime algorithm to compute high-quality triples with up to 24 million digits.
5. We developed a conjecture that is the analogue of the abc-conjecture for the new DGM quality class (Conjecture 3.17) and proved that this conjecture is conditional on the Lenstra-Pomerance-Wagstaff Conjecture in Theorem 3.16.

We use the following notation throughout this paper. $\ln$ represents the natural logarithm, $\exp$ represents the exponential function, $\mathbb{N}$ represents the set of positive integers, and $\mathbb{P}$ represents the set of prime numbers, $\mathbb{C}$ represents the set of complex numbers, and $a|b$ means that a divides b evenly. Sets are always in upper case, while sequences and variables are in lower case.

2 Background

2.1 Motivation for Quality Metric

We can motivate the idea of a quality metric of an abc-triple through Fermat's Last Theorem. Consider the nonlinear Diophantine equation

$$x^n + y^n = z^n \tag{1}$$

where $x, y, z, n \in \mathbb{Z}$. This equation has been shown to have infinitely many solutions when $n = 2$ [2]. Solutions where $x, y, z \in \mathbb{N}$ correspond to Pythagorean triples, and can be constructed as $(x, y, z) = (m(u^2 - v^2), 2uvm, m(u^2 + v^2))$ for $m \in \mathbb{N}$ and coprime positive integers u, v of opposite parity. However, when $n \geq 3$, Fermat's Last Theorem states that are no solutions to this equation.

Theorem 2.1 (Fermat's Last Theorem)**.** *There are no positive integral solutions in (x, y, z) to the equation $x^n + y^n = z^n$, where $n \geq 3, n \in \mathbb{N}$.*

Proof. The proof of this theorem can be found in Wiles' 1995 paper [14]. ☐

However, we can extend Fermat's Last Theorem from integers to polynomials. We replace x, y, z in Equation 1 with polynomials $x(t), y(t), z(t)$ in $\mathbb{C}[t]$ (the space of polynomials in t with complex coefficients). Consider the extension of Fermat's Last Theorem to this polynomial equation. It can be proved that there are no solutions to this polynomial equation as long as x, y, z have no common roots (that is, they do not share any linear factor $x - g$, where $g \in \mathbb{C}$). It is easier to prove this analogue of Fermat's Last Theorem than it is to prove the original theorem.

Theorem 2.2 (Fermat's Last Theorem for Polynomials)**.** *Let $n \geq 3, n \in \mathbb{N}$. Then, there are no solutions $x(t), y(t), z(t) \in \mathbb{C}[t]$ to the Fermat equation $[x(t)]^n + [y(t)]^n = [z(t)]^n$ where $x(t), y(t), z(t)$ do not share any common roots and are nonconstant.*

Proof. See [9] for proof. □

We can ignore n in the Fermat equation 1, and look at the polynomials $x(t), y(t), z(t)$ in $\mathbb{C}[t]$ satisfying

$$x(t) + y(t) = z(t) \tag{2}$$

Richard C. Mason [16] and Walter Wilson Stothers [17] independently proved the following theorem concerning solutions to Equation 2.

Theorem 2.3 (Mason-Stothers Theorem)**.** *If $x(t), y(t)$, and $z(t)$ are nonconstant polynomials with no common roots, and $x(t) + y(t) = z(t)$, then $\max\{\deg x(t), \deg y(t), \deg z(t)\} < \Big|\{\mu \in \mathbb{C} : (xyz)(\mu) = 0\}\Big|$. That is, the largest degree of either $x(t), y(t), z(t)$ is less than the number of distinct roots of $(xyz)(t)$.*

Proof. See [9], [16], or [17]. □

Remark. *In this theorem, the roots of $(xyz)(t)$ are being counted without multiplicity, which is why the cardinality of the set $\{\mu \in \mathbb{C} : (xyz)(\mu) = 0\}$ is not always $\deg xyz(t)$.*

Analogue of Mason-Stothers Theorem to Integers.

First, we will relate the polynomials $(x(t), y(t), z(t))$ in the Mason-Stothers Theorem to an abc-triple of integers (a, b, c). Similar to $x(t), y(t), z(t)$ having no common *roots*, we require that a, b, c have no common factors, and that (a, b, c) satisfy

$$a + b = c \tag{3}$$

just as $x(t) + y(t) = z(t)$.

A triple (a, b, c) that satisfies these conditions is designated an "abc-triple."

Definition 2.4 (abc-triple)**.** *A triple (a, b, c), where $a, b, c \in \mathbb{N}$, is called an abc-**triple** if and only if:*

1. $a + b = c$
2. $\gcd(a, b) = \gcd(b, c) = \gcd(a, c) = 1$

Next, we relate the roots of the polynomials $(x(t), y(t), z(t))$ to the radical of abc, and relate the degree of $(x(t), y(t), z(t))$ to the maximum value of (a, b, c).

Theorem 2.3 involves the number of distinct **roots** of the polynomial $(xyz)(t)$. Since distinct roots of a polynomial behave similarly to the *prime factors* of an integer, the proper analogue of the roots of $(xyz)(t)$ is the product of prime factors dividing abc, which is the radical of abc:

Definition 2.5 (Radical Function)**.** *The **radical** of a positive integer n is the product of the primes dividing n:*

$$\mathrm{rad}(n) = \prod_{p|n, p \in \mathbb{P}} p$$

The other side of the inequality in the conclusion of the Mason-Stothers involves the maximal degree of $x(t), y(t), z(t)$. The analogue of the *degree* in the integers is simply the absolute value. The analogue of $\max\{\deg x(t), \deg y(t), \deg z(t)\}$ is $\max\{|a|, |b|, |c|\}$. However, we will restrict triples to the *positive* integers for simplicity. Thus, since $a + b = c$ and $a, b, c > 0$, we have $\max\{|a|, |b|, |c|\} = \max\{a, b, c\} = c$.

Thus, the integer analogue of Mason-Stothers theorem will require us to compare the size of c to the size of $\mathrm{rad}(abc)$. The most natural comparison of these two quantities is to simply divide them; thus, we arrive at one possible formulation of the "quality" of an abc-triple:

Definition 2.6 (Test Quality)**.** *Let (a, b, c) be an abc-triple. Then, the "test quality" q_{test} of (a, b, c) is*

$$q_{test}(a, b, c) = \frac{c}{\mathrm{rad}(abc)}$$

Remark. *Triples with high test quality are considered important, because they have a **small** number of prime factors for their size. Such triples are said to be **round**. Thus, the quality is a way to search for round triples.*

It turns out that this "test quality" is not as interesting as the other quality metrics we will define below, as we can easily make it fairly large. There are a large number of triples (a, b, c) that have $q_{\text{test}}(a, b, c) > 1$. Given any fixed positive integer n_0, we can construct infinitely many abc-triples with test quality greater than any fixed positive integer n_0 using a certain family of triples ($a = 1, b = 2^{p(p-1)n} - 1, c = 2^{p(p-1)n}$).

Theorem 2.7. *Given any $c_0 \in \mathbb{N}$, let $S_{test} = \{q_{test}(a, b, c_0) : (a, b, c_0)$ is an abc-triple$\}$, and let $s_{c_0} = \max(S_{test})$. Then,*

$$\limsup_{c_0 \to \infty} s_{c_0} = \infty$$

Proof. Let n_0 be a positive integer. We will show that there are infinitely many triples with test quality greater than n_0, which is sufficient to prove the theorem. Let $p = 2n_0$ such that $p \in \mathbb{P}$. Consider the family of triples $(a, b, c) = (1, 2^{p(p-1)n} - 1, 2^{p(p-1)n})$, where n is any positive integer. Then, it can be shown that $\text{rad}(abc) < \frac{2c}{p}$ using Fermat's Little Theorem (see [15]).

Thus, for every triple in this family, we have

$$\begin{aligned} q_{\text{test}}(a, b, c) &= \frac{c}{\text{rad}(abc)} \\ &> \frac{c}{\frac{2c}{p}} \\ &= \frac{p}{2} \\ &= n_0 \end{aligned}$$

Thus, for any triple (a, b, c) in the family of triples that we've constructed, we have $q_{\text{test}}(a, b, c) > n_0$. Hence, there are infinitely many triples with quality greater than n_0.

□

Hence, by Theorem 2.7, the test quality is not a great quality metric to analyze the behavior of abc-triples since it is quite easy to make the test quality large. Instead, we would like to make it *difficult* to find high-quality triples. A quality metric that is more likely to be bounded than unbounded above is also more likely to be interesting. To accomplish this, it is better to tame the growth of the test quality by taking the natural logarithm of the numerator and denominator.

2.2 Standard Quality

Definition 2.8 (Standard Quality). *For a given abc triple (a, b, c), the standard quality, q_s, is:*

$$q_s(a, b, c) = \frac{\ln(c)}{\ln(\text{rad}(abc))}$$

Remark on Standard Quality. *We designate this quality as the "Standard Quality" because it has been analyzed extensively ([1, 4, 8, 13]) and is an elegant way to judge the roundness of an abc-triple. We will compare all of the alternate quality metrics that we define against this standard quality. Furthermore, it is critical that we restrict ourselves to only evaluating the quality on abc-triples (where a, b, c are coprime). Otherwise, we can trivially construct triples like $(a, b, c) = (2^h, 2^h, 2^{h+1})$ that have extremely small radical and, consequently, significantly high quality.*

A lot of analysis has already been conducted on the standard quality q_s. The values of q_s fluctuate quite a bit, and tend to be significantly unpredictable in their asymptotic behavior. It is known that there are infinitely many values of q_s that are greater than 1, but, apart from that, the best known asymptotic result comparing c to $\text{rad}(abc)$ is as follows.

a	b	c	**Quality** q_s	**Creator**
2	$3^{10} \cdot 109$	23^5	1.6299	Eric Ressyat [1]
11^2	$3^2 \cdot 5^6 \cdot 7^3$	$2^{21} \cdot 23$	1.6260	Benne de Weger [19]
$19 \cdot 1307$	$7 \cdot 29^2 \cdot 31^8$	$2^8 \cdot 3^{22} \cdot 5^4$	1.6235	Browkin, Brzezinski [18]
283	$5^{11} \cdot 13^2$	$2^8 \cdot 3^8 \cdot 17^3$	1.5808	Browkin, Brzezinski, and Nitaj [6, 18]
1	$2 \cdot 3^7$	$5^4 \cdot 7$	1.5679	Benne de Weger [19]
7^3	3^{10}	$2^{11} \cdot 29$	1.5471	Benne de Weger [19]
$7^2 \cdot 41^2 \cdot 311^3$	$11^{16} \cdot 13^2 \cdot 79$	$2 \cdot 3^3 \cdot 5^{23} \cdot 953$	1.5444	Abdherramane Nitaj [6]

Table 1: Some of the highest known values of q_s

Theorem 2.9 (Stewart and Yu, 2001)**.** *There is an absolute constant C_0 satisfying*

$$c < \exp\left(C_0 \cdot \sqrt[3]{\mathrm{rad}(abc)} \cdot (\ln(\mathrm{rad}(abc)))^3\right)$$

*for **all** abc-triples (a, b, c).*

Proof. See [12] for proof. □

On the computational side, there have been various algorithms (using distributed computing searches and advanced strategies such as continued fractions, elliptic curves, etc.) to find high quality triples [4, 5, 6, 7, 8, 13]. These computer searches are certainly not the central goal of the project, but we will use similar methods when searching for high values of our *new* qualities.

Table 1 categorizes high-quality triples that have been found using different computing approaches. Note that there are some triples that are rounder than others (in the sense that a round number has few prime factors for its size). In particular, the triple with fifth-highest quality, discovered by Benne de Weger, and the triple with highest quality, are more elegant than the 2nd and 3rd ranking triples [19]. This is simply because these elegant triples use really few primes in the factorizations of a, b, and c. This is the motivation for our DGM quality in Section 3, where we push this observation further by penalizing triples that use many primes in the factorization of abc.

Only 241 triples with standard quality values bigger than 1.4 have been found, so such triples are quite rare [4]. Despite extensive research on the Standard Quality, no triple with quality greater than 1.63 has been found. This is because the highest-quality triple known satisfies satisfies $q_s \approx 1.6299$, found by Ressyat (see [1]). This triple was calculated using the particularly special continued fraction expansion of $\sqrt[5]{109}$ [8].

Now, we will introduce a bit of the terminology that we will use throughout this paper. We would like to quantify when a particular (a, b, c) triple is noteworthy due to its high quality. We use the designation of a "hit" and a "high-quality triple" to create standards for such triples, as follows.

Definition 2.10 (abc-hit)**.** *A an abc-triple (a, b, c) is designated an "abc-hit" if $q_s(a, b, c) > 1$. Equivalently, a triple is an abc-hit if and only if $\mathrm{rad}(abc) < c$.*

Definition 2.11 (High-Quality Triple)**.** *An abc-triple (a, b, c) is designated a "high-quality triple" (also known as "good" abc-triples [4]) if $q_s(a, b, c) > 1.4$, where q_s is the Standard Quality.*

Similar definitions have been used by others [1, 4].

2.3 abc-conjecture

In this section, we use the standard quality to motivate and state the abc-conjecture. We also state several alternate forms of the conjecture. To begin, we will prove that there are infinitely many quality values greater than one. This follows almost directly from Theorem 2.7.

Theorem 2.12. *If $q_s(a, b, c)$ is the standard quality, then there are infinitely many abc-triples abc with standard quality greater than 1.*

Proof. Just as in the proof of Theorem 2.7, we consider an infinite family of triples that allows us to prove the theorem. If p is a prime greater than 2, and $n \in \mathbb{N}$, then the triple $(a, b, c) = a = 1, b = 2^{p(p-1)n} - 1, c = 2^{p(p-1)n}$ satisfies

$$\text{rad}(abc) < \frac{2c}{p}$$

This implies that

$$q_s(a,b,c) = \frac{\ln(c)}{\ln(\text{rad}(abc))} > \frac{\ln(c)}{\ln(\frac{2c}{p})}$$

Since $p > 2$, we know that $pc > 2c$ and hence $c > \frac{2c}{p}$. Note $f(x) = \ln(x)$ is monotonically increasing on the interval $x \in (0, \infty)$ since the derivative of $\ln(x)$ is always positive for $x \in (0, \infty)$. Thus,

$$c > \frac{2c}{p} \implies \ln(c) > \ln\left(\frac{2c}{p}\right) \implies \frac{\ln c}{\ln(\frac{2c}{p})} > 1 \implies q_s(a,b,c) > 1$$

Thus, $q_s(a,b,c) > 1$, for all possible a, b, c that constructed using this family of triples. Since there are infinitely many choices for p, n, there are infinitely many triples with quality higher than 1.

□

This theorem has a simple yet important corollary:

Corollary 2.12.1. *Given any fixed $c_0 \in \mathbb{N}$, we define $S_{std} = \{q_s(a,b,c_0) : (a,b,c_0)$ is an $abc-$ triple$\}$, and let $d_{c_0} = \max(S_{std})$. Then, if the limit superior of d_{c_0} exists, then*

$$\limsup_{c_0 \to \infty} d_{c_0} \geq 1$$

Proof. For any fixed $n_0 \in \mathbb{N}$, we know by Theorem 2.12 that there are infinitely many abc-triples (a,b,c) with $c > n_0$ and $q_s(a,b,c)$ greater than 1, so

$$\sup_{m \geq n_0} d_m \geq 1$$

Thus, we know that

$$\limsup_{c_0 \to \infty} d_{c_0} = \lim_{c_0 \to \infty} \sup_{m \geq c_0} d_m \geq \lim_{c_0 \to \infty} (1) = 1$$

by the squeeze theorem. □

However, after seeing that there are infinitely many abc-triples with quality greater than 1, one might wonder whether there are only *finitely* many triples with quality greater than 1.1, or 1.01, or 1.00001. We can generalize these statements by asking whether there are finitely many triples with quality greater than $1 + \epsilon$, for any real $\epsilon > 0$. This is the basis for Oesterlé and Masser's famous abc-conjecture [10, 11]:

Conjecture 2.13 (abc-conjecture - Oesterlé and Masser)**.** *For any real $\epsilon > 0$, there exist only finitely many abc-triples a, b, c such that $q_s(a,b,c) > 1 + \epsilon$.*

From large amounts of data collected, it appears that the conjecture might be true. However, we are still far from proving it as the closest result is far from an explicit bound on q_s (see Theorem 2.9).

Using Definition 2.8, we can develop an alternate form of the abc-conjecture as follows:

Conjecture 2.14 (abc-conjecture, alternate form)**.** *For any positive $\epsilon > 0$, there exist only finitely many abc-triples (a,b,c) such that $c > \Big(\text{rad}(abc)\Big)^{1+\epsilon}$.*

Another version of this abc-conjecture can be formulated in the spirit of Corollary 2.12.1. This version changes the inequality in the Corollary to an exact *equality*:

Conjecture 2.15 (abc-conjecture, alternate form 2)**.** *Let* $c_0 \in \mathbb{N}$, $S_{std} = \{q_s(a,b,c_0) : (a,b,c_0)$ *is an* $abc-$ *triple*$\}$, *and* $d_{c_0} = \max(S_{std})$. *Then,*

$$\limsup_{c_0 \to \infty} d_{c_0} = 1$$

Fermat's Last Theorem.

If we let $\epsilon = 1$ in the original abc-conjecture (Conjecture 2.13) then the conjecture states that there are only finitely many abc-triples with quality greater than 2. However, there have been *no* abc-triples found with $q_s(a,b,c) > 2$. This observation leads to a slightly refined version of the abc-conjecture for the special case $\epsilon = 1$:

Conjecture 2.16 (abc-conjecture, $\epsilon = 1$)**.** *There exist* ***no*** *abc-triples* (a,b,c) *such that* $q_s(a,b,c) > 2$.

This version of the conjecture is important because, if proven, it would imply a proof of Fermat's Last Theorem (Theorem 2.1) for $n \geq 6$. The cases $n = 3, 4, 5$ have already been proven independently and are well-known.

Theorem 2.17. *Assuming conjecture 2.16, the strong version of the abc-conjecture, is true, Fermat's Last Theorem (2.1) is true for* $n \geq 6$.

Proof. Following [1], let n be a positive integer with $n \geq 6$. Suppose Conjecture 2.16 is true. Then, we will show that Fermat's Last Theorem is also true using contradiction.

We assume that Fermat's Last Theorem is false for some $n \geq 6$. That is, there exist some $n, x, y, z \in \mathbb{Z}$ with $n \geq 6$ and $x^n + y^n = z^n$.

This statement implies that there exist some $n, x_0, y_0, z_0 \in \mathbb{Z}, n \geq 6$ with $x_0^n + y_0^n = z_0^n$ and x_0, y_0, z_0 are coprime. This is because if x, y, z satisfy $x^n + y^n = z^n$ and any *two* of x, y, z are not coprime, then all of x, y, z must have a common factor d greater than 1. This means that we can divide out by the common factor d to obtain a "primitive" solution x_0, y_0, z_0 with x_0, y_0, z_0 all coprime.

Since x_0, y_0, z_0 are coprime, the integers $a = x_0^n, b = y_0^n, c = z_0^n$ form an abc-triple. Now, $\text{rad}(abc)$ must exactly be equal to $\text{rad}(x_0 y_0 z_0)$. Due to multiplicativity of the radical function, $\text{rad}(x_0 y_0 z_0) = \text{rad}(x_0) \cdot \text{rad}(y_0) \cdot \text{rad}(z_0)$. Furthermore, $\text{rad}(n) \leq n$ for all $n \in \mathbb{N}$, so $\text{rad}(x_0 y_0 z_0) < x_0 y_0 z_0$. Then, we have the following:

$$\begin{aligned}
q_s(a,b,c) &= \frac{\ln c}{\ln(\text{rad}(abc))} \\
&\geq \frac{\ln c}{\ln(\text{rad}(x_0 y_0 z_0))} \\
&\geq \frac{\ln c}{\ln(x_0 y_0 z_0)} \\
&= \frac{\ln z_0^n}{\ln(x_0 y_0 z_0)} \\
&= \frac{n \cdot \ln z_0}{\ln(x_0) + \ln(y_0) + \ln(z_0)}
\end{aligned}$$

Thus, we know that

$$\begin{aligned}
q_s(a,b,c) &\geq \frac{n \cdot \ln z_0}{\ln(z_0) + \ln(z_0) + \ln(z_0)} \\
&= \frac{n \cdot \ln z_0}{3 \ln(z_0)} \\
&= \frac{n}{3}
\end{aligned}$$

Thus, falsity of Fermat's Last Theorem would imply that the quality is greater than $\frac{n}{3}$, which is greater than $\frac{6}{3} = 2$. However, this contradicts Conjecture 2.16. So Fermat's Last Theorem must be true by contradiction if Conjecture 2.16 is true. $\square$

This theorem illustrates just one of the consequences of the abc-conjecture. Other conjectures that would be resolved assuming the abc-conjecture are the Fermat-Catalan Conjecture and Szpiro's Conjecture [1].

We conclude this section with a plot, Figure 1, concerning the distribution of *high-quality triples* (triples with quality $q_s(a,b,c) \geq 1.4$) satisfying $c \leq 10^{18}$. This figure was created by Bart de Smit [4], and illustrates the complexity of the problem as well as the rarity of high-quality triples. The difficulty and unpredictability in creating high-quality triples can be seen from this figure, and our new quality metrics will have the same property.

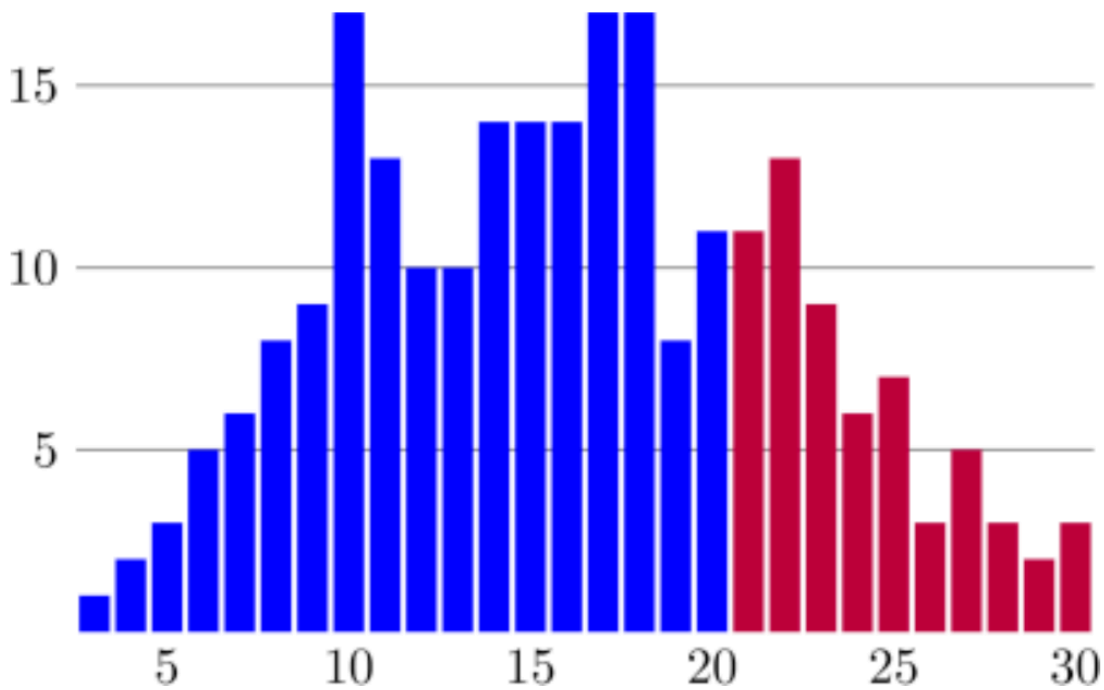


Figure 1: The distribution of high-quality abc-triples metricd over the number of digits of c [4] . The x-axis represents the number of digits of c, and the y-axis represents the number of High-Quality abc-triples with a certain number of digits.

2.4 Properties of Standard Quality

Now, we will manipulate the quality in several ways to obtain alternate, useful forms that we will refer to in future sections. We begin by expressing the quality using the prime factorization of the elements of a triple. Consider an abc-triple (a, b, c). Suppose that we use the Fundamental Theorem of Arithmetic to write the prime factorization of a as:

$$a = \prod_{i=0}^{k} p_i^{e_i}$$

where the p_i are primes, k is a positive integer, and the e_i are positive integers. Similarly, we can write b and c as

$$b = \prod_{i=k+1}^{m} p_i^{e_i}$$
$$c = \prod_{i=m+1}^{\omega} p_i^{e_i}$$

where m, ω are positive integers, and the sequences p, e are extended similarly (all the p_i remain primes and all the e_i remain positive integers). This notation allows us to write the prime factorization of abc as follows:

$$abc = \prod_{i=0}^{\omega} p_i^{e_i}$$

Note that the prime factorization of abc simply combines the prime factorizations of a, b, c since a, b, c are coprime. Here, ω is the number of primes dividing abc. In other words, if $S = \{p : p \in \mathbb{P}, p|abc\}$, then $|S| = \omega$. Now, we can write the quality in an alternate form:

$$\begin{aligned} q_s(a, b, c) &= \frac{\ln(c)}{\ln(\mathrm{rad}(abc))} \\ &= \frac{\ln\left(\prod_{i=m+1}^{\omega} p_i^{e_i}\right)}{\ln\left(\mathrm{rad}\left(\prod_{i=1}^{\omega} p_i^{e_i}\right)\right)} \end{aligned}$$

Expanding this form using the prime factorizations of a, b, c, we have

$$q_s(a, b, c) = \frac{\ln\left(\prod_{i=m+1}^{n=\omega} p_i^{e_i}\right)}{\ln\left(\mathrm{rad}\left(\prod_{i=1}^{k} p_i^{e_i} \cdot \prod_{i=k+1}^{m} p_i^{e_i} \cdot \prod_{i=m+1}^{\omega} p_i^{e_i}\right)\right)}$$

Using the definition of the radical (2.5), we can simplify the denominator of this fraction:

$$\begin{aligned} q_s(a, b, c) &= \frac{\ln\left(\prod_{i=m+1}^{\omega} p_i^{e_i}\right)}{\ln\left(\prod_{i=1}^{\omega} p_i\right)} \\ &= \frac{\sum_{i=m+1}^{\omega} \ln\left(p_i^{e_i}\right)}{\sum_{i=1}^{\omega} \ln(p_i)} \end{aligned}$$

Thus,

$$q_s(a, b, c) = \frac{\sum_{i=m+1}^{\omega} e_i \ln(p_i)}{\sum_{i=1}^{\omega} \ln(p_i)} \tag{4}$$

This form is immediately useful because it allows us to efficiently calculate the quality given the prime factorization of abc. However, in cases when abc is large, it is difficult to compute the quality - as there is no known polynomial-time prime factorization algorithm.

Equation 4 can also be extended further to include the concept of the Geometric Mean. This idea will be especially useful in Section 3. This idea introduces some immediate extra complexity to the way that the quality metric is written, but this complexity will simplify and is quite useful for our purposes.

By Equation 4, we know that

$$\begin{aligned} q_s(a,b,c) &= \frac{\ln(c)}{\sum_{p|abc}\ln(p)} \\ &= \frac{\ln(c)}{\ln\left(\prod_{i=1}^{\omega} p_i\right)} \\ &= \frac{\ln(c)}{\ln\left(\prod_{i=1}^{\omega}(p_i^{\omega})^{\frac{1}{\omega}}\right)} \end{aligned}$$

so

$$q_s(a,b,c) = \frac{\ln(c)}{\ln\left(\sqrt[w]{\prod_{i=1}^{\omega}(p_i^{\omega})}\right)} \tag{5}$$

The reason we write the quality this way is because the ωth root of the product of ω positive numbers is exactly the geometric mean of those numbers, which we will write as GM. Thus, we can write the quality as

$$\begin{aligned} q_s(a,b,c) &= \frac{\ln(c)}{\ln\left(GM\{p_1^{\omega}, p_2^{\omega}, ..., p_{\omega}^{\omega}\}\right)} \\ &= \frac{\ln(c)}{\ln(GM\{p_i^{\omega}\})} \end{aligned}$$

Note that the p_i^{ω} simplifies the way that we write the quality to characterize all possible values of p_i from $i = 1$ to ω. Hence, the quality itself is really a comparison of $\ln(c)$ to the Geometric Mean of the primes raised to the power of ω. In the following section, we change this geometric mean to a doubly geometric mean in order to ensure that the resulting high-quality triples use even fewer primes in their prime factorization.

At this point, we can use these representations of the standard quality to define new quality metrics and analyze these quality metrics in detail.

3 New Doubly Geometric Mean Quality

3.1 Motivation and Definition

This section defines a new quality metric, the Doubly Geometric Mean (DGM) Quality, which allows triples to be more round.

The third highest-quality triple (discovered by Browkin, Brzezinski), computed using the standard quality metric, is not quite as round as the other high-quality triples for q_s (see 1). This is because its prime factorization of uses several primes to moderately large powers. However, 1st, 5th, and 6th highest-quality triples can be considered more elegant because they all use few primes in their factorizations.

This is the motivation for us to define a new quality that privileges triples with a small number of primes in the prime factorizations of a, b, c (a small ω). The use of the radical in the standard quality achieves this to a certain extent - as shown in Table 1. Our new metric extends this idea even further. In particular, we can modify Equation 5 to enhance the influence of ω on the quality of a triple.

Our new quality should be easier to analyze, prefer round triples, and still satisfy the following properties:

1. The quality should be just as unpredictable as the standard quality in a qualitative sense.
2. For most triples, the quality values should lie within a small range. It should be rare to have a quality greater than 1 (DGM Hits: see Definition 3.5), and even rarer to have a quality greater than 1.4.
3. The quality should have comparable asymptotic behavior as the standard quality, and the resulting high-quality triples should be round.

4. The quality should still be a comparison of c to some function $\Delta(a,b,c)$, where $\Delta(a,b,c)$ can be expressed entirely in terms of the prime factorizations of a,b,c.

It is nontrivial to find a quality metric that satisfies all of this conditions due to the multiple requirements that it must satisfy. We decided to use the elegance of Equation 5 to define a quality in terms of the Doubly Geometric Mean. This quality satisfies all of the properties specified in the list above. To define this quality, we first need to consider the Doubly Geometric Mean (DGM).

Definition 3.1 (Doubly Geometric Mean)**.** *Let S be a set of positive real numbers: $S = \{x_1, x_2, x_3, \ldots, x_k\}$, where k is a positive integer and $k = |S|$. The Doubly Geometric Mean (or DGM) of S is:*

$$\mathit{DGM}(S) = \mathit{DGM}(\{x_1, x_2, ..., x_k\}) = \exp\left(\exp\left(\frac{1}{k}\sum_{i=1}^{k}\ln(\ln(x_i))\right)\right)$$

Using the DGM, we define the first new quality metric, the DGM Quality (or Doubly Geometric Mean Quality).

Definition 3.2 (DGM Quality)**.** *Let (a,b,c) be an abc-triple. Consider the sequence of primes dividing the product abc, (p_i). Let P be the set of all terms in this sequence ($P = \{p : p|abc, p \in \mathbb{P}\}$), and let $\omega = |P|$.*

Then, let $P_\omega = \{p^\omega : p \in P\}$. That is, P_ω is the set of primes dividing abc, all raised to the power of ω.

Then, the Doubly Geometric Mean Quality, or DGM Quality, is defined as

$$q_{\mathit{DGM}}(a,b,c) := \frac{\ln(c)}{\ln(\mathit{DGM}(P_\omega))} = \frac{\ln(c)}{\ln(\mathit{DGM}(\{p_1^\omega, p_2^\omega, ..., p_\omega^\omega\}))}$$

Remark on relationship to standard quality. *The standard quality was a comparison of $\ln(c)$ to the $\ln$ of the Geometric Mean (GM) of the primes raised to the power of ω, whereas this quality is a comparison of $\ln(c)$ to the $\ln$ of the Doubly Geometric Mean (DGM) of the primes raised to ω. This allows us to privilege triples that use a significantly small number of primes, almost regardless of what their size is.*

However, in order to understand the significance of using the DGM instead of the GM in this quality metric, we need to understand the behavior of the DGM. The definition of the DGM follows from the following sequence of means, from the Arithmetic Mean to the Geometric and Doubly Geometric Mean.

$$\begin{aligned}
AM(x_1, x_2, ..., x_k) &= \frac{1}{k}\sum_{i=1}^{k} x_i \\
GM(x_1, x_2, ..., x_k) &= \exp\left(\frac{1}{k}\sum_{i=1}^{k}\ln(x_i)\right) = \exp\Big(AM\big\{\ln(x_1), \ln(x_2), \ldots, \ln(x_k)\big\}\Big) \\
DGM(x_1, x_2, ..., x_k) &= \exp\left(\exp\left(\frac{1}{k}\sum_{i=1}^{k}\ln\ln(x_i)\right)\right) \\
&= \exp\left(\exp\Big(AM\big\{\ln(\ln(x_1)), \ln(\ln(x_2)), \ldots, ..., \ln(\ln(x_k))\big\}\Big)\right)
\end{aligned}$$

As we progress down the sequence of means, we add another $\exp$ on the outside of the sum and another $\ln$ on the inside of the sum. The $\ln$, for our purposes, serves to tame the large primes. The $\exp$ ensures that the AM, GM, DGM all grow at roughly comparable rates. For example, when all of the x_i are equal, we would like the means to have the exact same value, which can be accomplished through the addition of the outer $\exp$.

Using this idea, the DGM can be considered more extreme than the AM and the GM because it tames the values of the x_i (using two $\ln$s) before averaging them. The DGM is really a **tool** that allows us to emphasize the importance of using very few primes regardless of their size. This allows us to enhance the significance of ω in the quality metric irrespective of the size of the primes used.

It turns out that we can simplify the quality metric (Definition 3.2). To do so, we will rewrite q_{DGM} as follows:

$$q_{DGM}(a,b,c) = \frac{\ln(c)}{\ln\left(\exp\left(\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega}\ln\ln(p_i^{\omega})\right)\right)\right)}$$

(where (p_i) is the sequence of primes dividing abc)
Then, we can cancel the outside $\ln$ with the $\exp$, since $\ln(e^x) = x$:

$$q_{DGM}(a,b,c) = \frac{\ln(c)}{\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega}\ln\ln(p_i^{\omega})\right)}$$

Using rules of logarithms, we rewrite the inner $\ln$s of the denominator:

$$\begin{aligned} q_{DGM}(a,b,c) &= \frac{\ln(c)}{\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega}\ln(\omega\cdot\ln(p_i))\right)} \\ &= \frac{\ln(c)}{\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega}\left(\ln(\omega)+\ln(\ln(p_i))\right)\right)} \\ &= \frac{\ln(c)}{\exp\left(\frac{1}{\omega}\left(\omega\ln(\omega)+\sum_{i=1}^{\omega}\left(\ln(\ln(p_i))\right)\right)\right)} \\ &= \frac{\ln(c)}{\exp\left(\ln(\omega)+\frac{1}{\omega}\left(\sum_{i=1}^{\omega}\left(\ln(\ln(p_i))\right)\right)\right)} \end{aligned}$$

so we can simplify to obtain the following:

$$\begin{aligned} q_{DGM}(a,b,c) &= \frac{\ln(c)}{\omega\cdot\exp\left(\frac{1}{\omega}\left(\sum_{i=1}^{\omega}\ln(\ln(p_i))\right)\right)} \\ &= \frac{\ln(c)}{\omega\cdot\exp\left(\sum_{i=1}^{\omega}\ln(\ln(p_i))\right)^{\frac{1}{\omega}}} \\ &= \frac{\ln(c)}{\omega\cdot\sqrt[w]{\left(\prod_{i=1}^{\omega}\exp(\ln(\ln(p_i)))\right)}} \\ &= \frac{\ln(c)}{\omega\cdot\sqrt[w]{\left(\prod_{i=1}^{\omega}\ln(p_i)\right)}} \end{aligned}$$

Thus,

$$q_{DGM}(a,b,c) = \frac{\ln(c)}{\omega\cdot\sqrt[w]{\left(\prod_{i=1}^{\omega}\ln(p_i)\right)}} \tag{6}$$

Remark comparing Standard Quality to DGM Quality. *Compare this Equation 6 for the simplified version of q_{DGM} to Equation 4 for the simplified form of q_s. Both involve a geometric mean, but the standard quality version does not scale by $\frac{1}{\omega}$, and q_s raises the primes to the ω in the GM whereas q_{DGM} does not. Finally, the $\ln$ is on the inside of the GM for the DGM Quality, whereas it is on the outside for the Standard Quality.*

From this formula, we can make the following observations:

1. The value of ω is inversely proportional to the quality. Triples that use large values of ω (and, hence, use a large number of primes in their factorizations) will have their quality values directly reduced by a large factor.
2. This version of the quality is a comparison of c to the GM of the logarithms of the primes dividing abc (without raising them to the ω power, as in the definition of the DGM Quality). Thus, using this formula for the quality reduces the difficulty of computing quality values.
3. In this quality, we are focusing primarily on only *one* condition: a small value of ω. This allows us to isolate one variable and simplifies the difficulty of the problem.

3.2 Interesting Properties and High Qualities

We will analyze the behavior of this new quality metric. The first theorem that we can prove is a lower bound for q_{DGM}. To do this, we will need the following Lemma:

Lemma 3.3. *Let s be a fixed positive integer. Let (p_i) be the sequence of primes dividing s. Suppose that this sequence has length ω_s. Then,*

$$\sum_{i=0}^{\omega_s} \ln(p_i) \leq \ln(s)$$

Proof. Let the prime factorization of s be

$$s = \prod_{i=0}^{\omega_s} p_i^{e_i}$$

where (e_i) is a sequence of positive integers.

Then, taking the logarithms of both sides, we have

$$\ln(s) = \ln\left(\prod_{i=0}^{\omega_s} p_i^{e_i}\right)$$

$$\ln(s) = \sum_{i=0}^{\omega_s} e_i \ln(p_i)$$

Since all of the $e_i \geq 1$, we have

$$\sum_{i=0}^{\omega_s} e_i \ln(p_i) \geq \sum_{i=0}^{\omega_s} \ln(p_i)$$

so

$$\sum_{i=0}^{\omega_s} \ln(p_i) \leq \ln(s)$$

with equality occurring exactly when s is squarefree.

□

This Lemma allows us to prove the following theorem about a lower bound for q_{DGM}.

Theorem 3.4. *For every abc-triple (a, b, c), we have $q_{DGM}(a, b, c) > \frac{1}{3}$.*

Proof. Let (a, b, c) be an abc-triple.

We know that

$$q_{\text{DGM}}(a, b, c) = \frac{\ln(c)}{\omega \cdot \sqrt[w]{\left(\prod_{i=1}^{\omega} \ln(p_i)\right)}}$$

By the arithmetic-geometric mean inequality, we have

$$\begin{aligned} q_{\text{DGM}}(a, b, c) &\geq \frac{\ln(c)}{\omega \cdot \frac{1}{\omega} \sum_{i=1}^{\omega} \ln(p_i)} \\ &= \frac{\ln(c)}{\sum_{i=1}^{\omega} \ln(p_i)} \end{aligned}$$

By Lemma 3.3, we obtain

$$\begin{aligned} q_{\text{DGM}}(a, b, c) &\geq \frac{\ln(c)}{\omega \cdot \frac{1}{\omega} \sum_{i=1}^{\omega} \ln(p_i)} \\ &= \frac{\ln(c)}{\sum_{p|abc} \ln(p)} \\ &\geq \frac{\ln(c)}{\ln(abc)} \\ &\geq \frac{\ln(c)}{\ln(c^3)} \\ &= \frac{1}{3} \end{aligned}$$

□

Now that we have a lower bound for the lowest-quality triples, we investigate the triples with high quality.

We define certain standards (similar to Definitions 2.10 and 2.11) to judge whether a certain triple has high DGM Quality.

Definition 3.5 (DGM Hit)**.** *An abc-triple (a, b, c) is designated a "DGM hit" if $q_{DGM}(a, b, c) > 1$. That is, a triple in a DGM hit if and only if $DGM(P_\omega) < c$.*

Definition 3.6 (High-Quality DGM hit)**.** *An abc-triple is a "High-Quality DGM hit" if it has DGM Quality greater than 5 - that is, $q_{DGM}(a, b, c) > 5$.*

Remark on Definitions 3.5 and 3.6. *The lower limit of $q_{DGM}(a, b, c) = 5$ for a High-Quality DGM Hit is stricter than the lower limit for a High-Quality abc-hit (which is $q_s(a, b, c) = 1.4$). This distinction is due to the DGM Quality growing faster than the Standard Quality.*

Qualitative Analysis of DGM Quality on Small abc-triples.

Figure 2 shows the DGM Quality evaluated on *all* abc-triples (a, b, c) satisfying $c \leq 100$. This figure illustrates the unpredictability of the quality values. Notice that several of the quality values lie between 0.5 and 0.7, as desired. The red x's in the figure correspond to triples that are DGM hits, and the x's correspond to abc-triples (a_0, b_0, c) that have quality higher than any other abc-triple (a_1, b_1, c) (they are the *best* triples for their size). We would like to develop algorithms to search for the red x's and high-quality DGM triples.

In order to categorize these high-quality triples, we define the sequence s_c as follows:

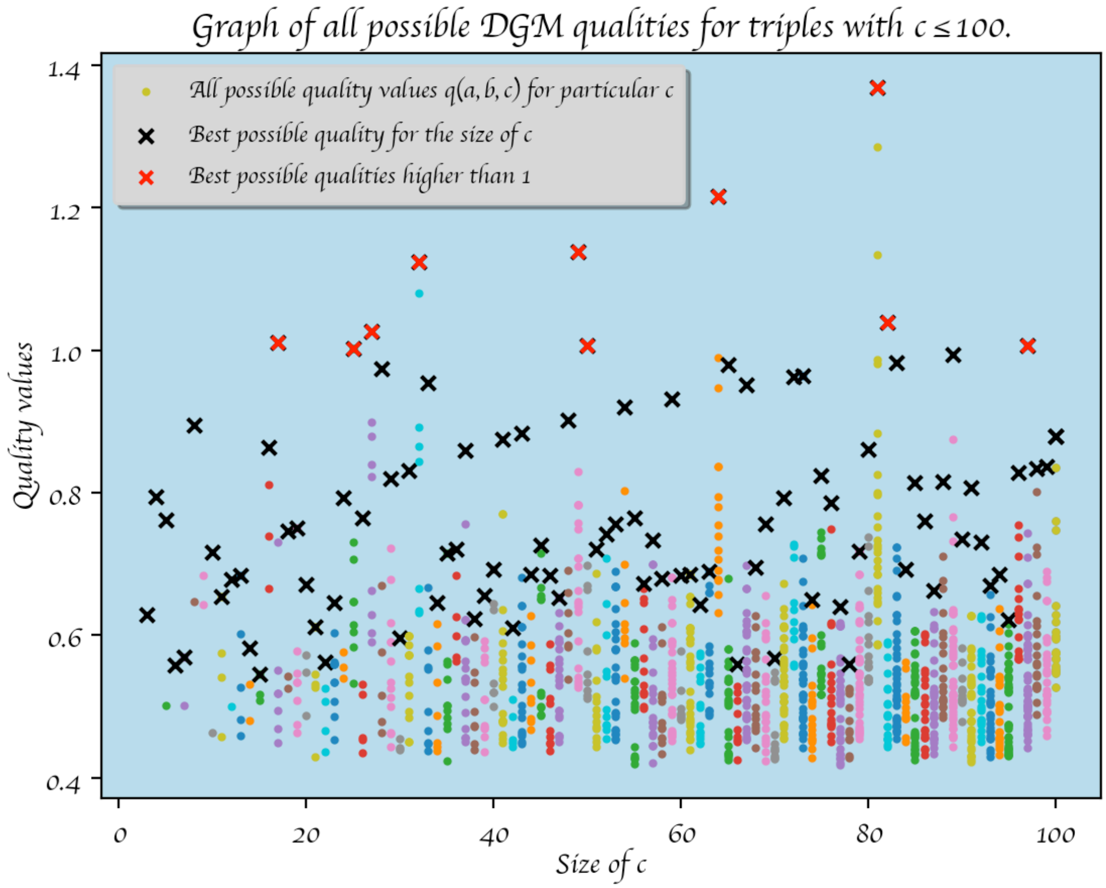


Figure 2: The quality evaluated on all abc-triples with $c \leq 100$. Red x's are DGM hits, and black x's correspond to triples with highest quality for a given c.

Definition 3.7. *Let $S_c = \{q_{DGM}(a, b, c) : (a, b, c)$ is an abc-triple$\}$. Then, $s_c = \max S_c$ for any positive integer c. s_c is the DGM analogue for d_{c_0}, which involves the standard quality.*

We would like to investigate the asymptotic behavior of s_c. In particular, we are looking for families of triples for which s_c is expected to be quite high. This is the focus of the following section.

3.3 Families of High-Quality Triples

We will define five constructions of triples, which all produce high quality triples for q_{DGM}.

3.3.1 Power of 2, 3 Method

The simplest possible case of such a construction is when a is a power of 2, and b is a power of 3. That is, let i, j be positive integers such that

$$a = 2^i$$
$$b = 3^j$$

Let $c = a + b$. Note that ω is simply the number of primes dividing c, plus two. This means that, if c uses very few prime factors, then (a, b, c) will have large DGM Quality.

The simplest case of this "Power of 2, 3" construction is when c itself is a prime. Then, we know that ω is exactly 3. So, by the simplification of q_{DGM},

$$\begin{aligned}
q_{DGM}(2^i, 3^j, c) &= \frac{\ln(c)}{3\sqrt[3]{\prod_{i=1}^{3}(\ln(p_i))}} \\
&= \frac{\ln(c)}{3 \cdot \sqrt[3]{\ln(2)\cdot\ln(3)\cdot\ln(c)}} \\
&= (\ln(c))^{\frac{2}{3}} \cdot \frac{1}{3\sqrt[3]{\ln(2)\cdot\ln(3)}} \\
&\approx 0.365 \cdot (\ln(c))^{\frac{2}{3}}
\end{aligned}$$

so

$$q_{DGM}(2^i, 3^j, c) > \frac{1}{3}(\ln(c))^{\frac{2}{3}} \tag{7}$$

which goes to infinity (albeit slowly) as $c \to \infty$. This observation allows us to prove the following theorem.

Theorem 3.8. *Consider $i, j \in \mathbb{N}$, and $a = 2^i, b = 3^j, c = a + b$. If there exist infinitely many pairs (i, j) such that c is prime, then we have*

$$\limsup_{c\to\infty} s_c = \infty$$

Proof. Assume that there exist infinitely many pairs i, j such that $c = 2^i + 3^j$ and c is prime. Then, we can make c arbitrarily large. Now, let N be a positive integer. By assumption, there must exist some c_0 with $c_0 > \exp(3N\sqrt{3N})$. For all abc-triples $(a, b, c) = (2^i, 3^j, 2^i + 3^j)$ with $c = 2^i + 3^j > c_0$, we know, by Equation 7, that $q_{DGM}(a, b, c) > (\ln(c))^{\frac{2}{3}}$. Hence, since $c > c_0$, we have

$$\begin{aligned}
q_{DGM}(2^i, 3^j, c) &> \frac{1}{3}(\ln(c))^{\frac{2}{3}} \\
&> (\ln(c_0))^{\frac{2}{3}} \\
&> \frac{1}{3}(\ln(\exp(3^{\frac{3}{2}}N\sqrt{N})))^{\frac{2}{3}} \\
&= \frac{1}{3}(3^{\frac{3}{2}}N\sqrt{N})^{\frac{2}{3}} \\
&= \frac{1}{3}\cdot 3 \cdot N^{\frac{2}{3}\cdot\frac{3}{2}} \\
&= N
\end{aligned}$$

Thus, for any positive integer N, there exist infinitely many triples with quality greater than N. Thus, it follows immediately that

$$\limsup_{c\to\infty} s_c = \infty$$

$\square$

So, having a being a power of 2 and b being a power of 3 can result in quickly growing quality values (qualities that grow in logarithmic fashion). We will utilize this process later in our algorithms section, Section 6.

3.3.2 Power of p, q method

We can generalize the Power of 2, 3 approach to using any two primes p, q, not just 2 and 3.

Theorem 3.9. *Let p, q be distinct primes. Let i, j be positive integers such that $a = p^i, b = q^j, c = a + b$. If there exist infinitely many pairs (i, j) such that $c = p^i + q^j \in \mathbb{P}$, then*

$$\limsup_{c\to\infty} s_c = \infty$$

Proof. As in the proof of Theorem 3.8, assume that there exist infinitely many pairs i, j such that $c = p^i + q^j$ is prime.

Let N be a positive integer. We will show that there are infinitely many triples with quality greater than N, using our construction.

There must exist some $c_0 = p^i + q^j \in \mathbb{P}$ with $c_0 > \exp\left(\left(N \cdot \sqrt[3]{3\ln(p)\ln(q)}\right)^{\frac{3}{2}}\right)$. So, for all abc-triples $(a, b, c) = (p^i, q^j, p^i + q^j)$ with $c = p^i + q^j > c_0$ and c prime, we know $\omega = 3$ and

$$\begin{aligned}
q_{DGM}(p^i, q^j, c) &= \frac{\ln(c)}{3\sqrt[3]{\prod_{i=1}^{3}(\ln(p_i))}} \\
&= \frac{\ln(c)}{3 \cdot \sqrt[3]{\ln(p) \cdot \ln(q) \cdot \ln(c)}} \\
&= (\ln(c))^{\frac{2}{3}} \cdot \frac{1}{\sqrt[3]{3 \cdot \ln(p) \cdot \ln(q)}}
\end{aligned}$$

Then,

$$\begin{aligned}
q_{DGM}(p^i, q^j, c) &= (\ln(c))^{\frac{2}{3}} \cdot \frac{1}{\sqrt[3]{3 \cdot \ln(p) \cdot \ln(q)}} \\
&> (\ln(c_0))^{\frac{2}{3}} \cdot \frac{1}{\sqrt[3]{3 \cdot \ln(p) \cdot \ln(q)}} \\
&> \left(\ln\left(\exp\left(\left(N \cdot \sqrt[3]{3\ln(p)\ln(q)}\right)^{\frac{3}{2}}\right)\right)\right)^{\frac{2}{3}} \cdot \frac{1}{\sqrt[3]{3 \cdot \ln(p) \cdot \ln(q)}} \\
&= \left(\left(N \cdot \sqrt[3]{3\ln(p)\ln(q)}\right)^{\frac{3}{2}}\right)^{\frac{2}{3}} \cdot \frac{1}{\sqrt[3]{3 \cdot \ln(p) \cdot \ln(q)}} \\
&= N
\end{aligned}$$

Thus, for any positive integer N, there exist infinitely many triples with quality greater than N, and

$$\limsup_{c\to\infty} s_c = \infty$$

and we are done. □

3.3.3 Fixed Sequence of Primes Method

In creating high-quality triples (a, b, c), a and b need not be perfect prime powers. For example, if a can be expressed as a product of powers of 2 and 3, and b can be expressed as a product of powers of 5, 7 and 11, then the DGM quality can still be made quite large. In fact, we can prove the following theorem that shows that the qualities will eventually get large if c is prime and a, b can always be expressed using any finite, fixed product of prime powers.

Theorem 3.10. *Given a fixed sequence of primes $a_1, a_2, a_3, ..., a_n$, and another fixed sequence of primes $b_1, b_2, ..., b_m$, construct positive integers a, b by creating sequences of positive integers $i_1, i_2, i_3, ..., i_n$, $j_1, j_2, ..., j_m$, so that*

$$a = \prod_{k=0}^{n} a_k^{i_k}$$
$$b = \prod_{k=0}^{m} b_k^{j_k}$$

If there are infinitely many choices of the sequences $i_1, i_2, i_3, ..., i_n$, $j_1, j_2, ..., j_m$ such that $c = a + b$ is prime, we must have

$$\limsup_{c \to \infty} s_c = \infty$$

Proof. Suppose that there are infinitely many choices of the sequences $i_1, i_2, i_3, ..., i_n, j_1, j_2, ..., j_m$ such that $c = a+b$ is prime. Then, we have $\omega = m + n + 1$ and:

$$\begin{aligned}
q_{DGM}(a, b, c) &= \frac{\ln(c)}{\omega \cdot \sqrt[w]{\left(\prod_{i=0}^{\omega}(\ln(p_i)\right)}} \\
&= \frac{\ln(c)}{(m+n+1)\ \sqrt[m+n+1]{\left(\prod_{i=0}^{m+n+1}\ln(p_i)\right)}} \\
&= \frac{\ln(c)}{(m+n+1)\ \sqrt[m+n+1]{\left(\prod_{i=0}^{m+n}\ln(p_i)\right)\cdot \ln(c)}} \\
&= \ln(c)^{1-\frac{1}{m+n+1}} \cdot \frac{1}{\sqrt[m+n+1]{\prod_{i=0}^{m+n}\ln(p_i)}} \\
&= \ln(c)^{1-\frac{1}{m+n+1}} \cdot C
\end{aligned}$$

where C is a constant that does not depend on the value of c.

Now, since $\ln(x) \to \infty$ as $x \to \infty$, we know that

$$\begin{aligned}
\lim_{c\to\infty} \ln(c)^{1-\frac{1}{m+n+1}} \cdot C &= C \lim_{c\to\infty} \ln(c)^k \text{ where } k > 0 \\
&= \infty
\end{aligned}$$

Thus, for any positive integer N, there are infinitely many triples with DGM Quality greater than N, so we must have

$$\limsup_{c \to \infty} s_c = \infty$$

□

Remark on the Generality of this Approach. *This approach is a generalized version of the Power of $2, 3$ and p, q methods. For both of these methods, $m = n = 1$, since the prime factorization of a, b use only one prime.*

While this approach is quite powerful in proving that there are several high-quality triples, there are still challenges in *finding* such triples computationally. For example, we can find high-quality triples by choosing a and b to be the products of small primes raised to large powers, and check whether c is prime. However, it is difficult to determine whether c is prime or not, since this requires thousands of primality tests on large values of c. We need to use a simpler approach that is asymptotically more efficient than this one. Ideally, this method should allows us to restrict ourselves to an extremely small number of triples. The answer lies in Fermat and Mersenne primes, which are the foci of the next two sections.

3.3.4 Fermat Prime Method

We will begin by considering Fermat primes to create high-quality triples, which we call the Fermat Prime Method.

Definition 3.11 (Fermat Primes)**.** *Suppose that n is a positive integer such that $2^n + 1$ is prime. Then, $2^n + 1$ is called a Fermat prime and n is called a Fermat exponent.*

For $2^n + 1$ to be a Fermat Prime, it is well known that n itself must be a power of 2 [20]. Thus, all Fermat Primes are of the form $2^{2^k} + 1$, for some positive integer k.

We take advantage of their properties of Fermat Primes and construct a family of high-quality triples. Let n be a Fermat Exponent, and let $(a, b, c) = (1, 2^n, 2^n + 1)$. This creates an abc-triple, which we call a *"Fermat triple."* In this case, the number of primes used in the prime factorization of abc will be exactly 2, since $c \in \mathbb{P}$. Thus, ω is *always* 2 for any Fermat triple. Thus, Fermat triples will result in high-quality DGM triples, as we quantify in the following theorem.

Theorem 3.12. *Let (a, b, c) be a Fermat triple, with Fermat exponent n. Then the quality $q_{DGM}(a, b, c)$ must be greater than $\frac{\sqrt{n}}{2}$.*

Proof. Let $(a, b, c) = (1, 2^n, 2^n + 1)$, with Fermat exponent n. The sequence p_i of primes dividing abc is simply $p_1 = 2, p_2 = c$. Thus,

$$
\begin{aligned}
q_{DGM}(a, b, c) &= \frac{\ln(c)}{\omega \cdot \sqrt[w]{\left(\prod_{i=0}^{\omega}(\ln(p_i)\right)}} \\
&= \frac{\ln(c)}{2 \cdot \sqrt{\left(\prod_{i=1}^{2}(\ln(p_i)\right)}} \\
&= \frac{\ln(c)}{2 \cdot \sqrt{\ln(2) \cdot \ln(c)}}
\end{aligned}
$$

Simplifying, we have

$$
\begin{aligned}
q_{DGM}(a, b, c) &= \sqrt{\ln(c)} \cdot \frac{1}{2\sqrt{\ln(2)}} \\
&= \sqrt{\ln(c)} \cdot C
\end{aligned}
$$

so the quality is proportional to the square root of the logarithm of c. We can rewrite this expression in terms of n to obtain our desired result:

$$
\begin{aligned}
q_{DGM}(a, b, c) &= \sqrt{\ln(2^n + 1)} \cdot \frac{1}{2\sqrt{\ln(2)}} \\
&> \sqrt{\ln(2^n)} \cdot \frac{1}{2\sqrt{\ln(2)}} \\
&= \sqrt{n \cdot \ln(2)} \cdot \frac{1}{2\sqrt{\ln(2)}} \\
&= \frac{\sqrt{n}}{2}
\end{aligned}
$$

$\square$

A use of this theorem is that it allows us to make the DGM quality values quite large, as the following Corollary quantifies.

Corollary 3.12.1. *If there are infinitely many Fermat primes, then*

$$\limsup_{c\to\infty} s_c = \infty$$

Proof. Let N_0 be a fixed positive integer. Assume that there are infinitely many Fermat primes. Then, there are infinitely many Fermat abc-triples $(a,b,c) = (1, 2^n, 2^n+1)$ where $c \in \mathbb{P}$. All of these triples satisfy $q_{DGM}(a,b,c) > \frac{\sqrt{n}}{2}$ by Theorem 3.12. Thus, for all n with $n > (2N_0)^2$, we have $\frac{\sqrt{n}}{2} > N_0$ so $q_{DGM}(a,b,c) > N_0$. Hence, there are infinitely many triples with quality greater than N_0 if there are infinitely many Fermat primes. □

The disadvantage to using the Fermat prime strategy, however, is that there are only 5 known Fermat primes, the largest of which is $2^{16}+1 = 65537$. Thus, we can only use this approach to make the quality only slightly larger than $\frac{\sqrt{n}}{2} = \frac{\sqrt{16}}{2} = 2$. It is also conjectured (but not known) that there are *only* 5 such primes - in that case, it would be impossible to make the quality much larger than 2.

We still need a more efficient method to find high-quality triples. Instead of looking for primes of the form 2^n+1, we look for primes of the form 2^n-1, which are called Mersenne primes - the focus of the next section.

3.3.5 Mersenne Prime Method

We use a similar approach to the Fermat Prime Method to create a family of high-quality triples, but use Mersenne Primes instead.

Definition 3.13. *Let n be a natural number. If the number 2^n-1 is prime, then 2^n-1 is called a "Mersenne prime" and n is called a "Mersenne exponent."*

Remark. *It has been shown that all Mersenne exponents must be prime. That is, 2^n-1 can be prime only if n itself is prime [21]. This reduces the search space for Mersenne exponents from $\mathbb{N}$ to $\mathbb{P}$. There are 51 known Mersenne primes, and the largest Mersenne prime is, in fact, the largest prime number known [22]. This is because it is easier to determine primality through Mersenne primes than for most other prime numbers.*

Now, we define a Mersenne triple as follows: if 2^n-1 is a Mersenne prime, then we let $a = 1, b = 2^n-1, c = 2^n$. This *Mersenne triple* (a,b,c) is also an abc-triple, so we can evaluate the DGM quality on this triple. We find that the DGM quality on Mersenne triples has a similar behavior to the DGM quality on Fermat triples, as seen in the following theorem.

Theorem 3.14. *Let n be a Mersenne exponent. Then the Mersenne triple (a,b,c) given by $a = 1, b = 2^n-1, c = 2^n$ has DGM quality greater than $\frac{\sqrt{n}}{2}$.*

Proof. There are only two primes ($\omega = 2$) in the prime factorization of abc and these primes are $p_0 = 2, p_1 = c$. Thus, we know that

$$\begin{aligned}
q_{DGM}(a,b,c) &= \frac{\ln(c)}{\omega \cdot \sqrt[w]{\left(\prod_{i=0}^{\omega}(\ln(p_i)\right)}} \\
&= \frac{\ln(c)}{2 \cdot \sqrt{\left(\prod_{i=1}^{2}(\ln(p_i)\right)}} \\
&= \frac{\ln(c)}{2 \cdot \sqrt{\ln(2) \cdot \ln(c)}} \\
&= \sqrt{\ln(c)} \cdot \frac{1}{2\sqrt{\ln(2)}}
\end{aligned}$$

just as in the case of Fermat primes. This allows us to find, using the exact same reasoning as in the proof of Theorem 3.3, that

$$\begin{aligned} q_{DGM}(a,b,c) &= \sqrt{\ln(2^n)} \cdot \frac{1}{2\sqrt{\ln(2)}} \\ &= \sqrt{n \cdot \ln(2)} \cdot \frac{1}{2\sqrt{\ln(2)}} \\ &= \frac{\sqrt{n}}{2} \end{aligned}$$

□

Remark on DGM Quality of Fermat vs. Mersenne Triples *The Fermat triples are slightly better than the Mersenne triples because the quality for the Fermat triples is always slightly larger than* $\frac{\sqrt{n}}{2}$ *whereas the quality of Mersenne triples is always* $\frac{\sqrt{n}}{2}$.

Theorem 3.14 allows us to claim that Mersenne triples are, asymptotically, the same as Fermat triples. However, there are many more known Mersenne primes than there are Fermat primes - the largest known Mersenne prime is of the form $2^{82,589,933} - 1$, with Mersenne exponent $n = 82,589,933$ [22]. It is, in fact, conjectured that there are infinitely many Mersenne primes, and computer suggest that this could be true.

Using the , we find the high-quality triple

$$a = 1, b = 2^{82,589,933} - 1, c = 2^{82,589,933}$$

Using Theorem 3.14, we know that

$$\begin{aligned} q_{DGM}(a,b,c) &= \frac{\sqrt{n}}{2} \\ &= \frac{\sqrt{82,589,933}}{2} \\ &\approx 4543.95 \end{aligned}$$

This is the largest High-Quality DGM Triple that we found through any method.

Such a high quality triple suggests that the limit superior of s_c is ∞ (that is, the DGM quality could systematically be made as large as one pleases), but we were only able to prove this statement conditionally. For example, suppose that the following conjecture is true.

Conjecture 3.15 (Lenstra-Pomerance-Wagstaff Conjecture [23, 24]). *There are infinitely many Mersenne primes, and the number of Mersenne primes less than n is well approximated by* $\exp(\gamma) \cdot \log_2(\log_2(n))$, *where γ is the the Euler-Mascheroni Constant.*

The conjecture implies the following theorem.

Theorem 3.16. *If Conjecture 3.15 is true, then*

$$\limsup_{c \to \infty} s_c = \infty$$

Proof. Assuming Conjecture 3.15, there are infinitely many Mersenne primes and hence infinitely many Mersenne triples. Let N_0 be a fixed positive integer. Hence, there are infinitely many abc-triples of the form $(1, 2^n, 2^n + 1)$ where $2^n + 1 \in \mathbb{P}$, so we can make n arbitrarily large. By Theorem 3.14, every such triple has quality greater than $\frac{\sqrt{n}}{2}$, which we make larger than N_0 by setting $n = \lceil 4N_0^2 \rceil$. Thus, there are infinitely many triples with quality greater than N_0. □

However, an important corollary of this theorem is that if the maximum value of s_c was shown to be *finite*, then we can resolve Conjecture 3.15.

Corollary 3.16.1. *If s_c is bounded above, then Conjecture 3.15 is false and there are only finitely many Mersenne primes.*

Proof. The corollary follows directly from Theorem 3.16, as it is the contrapositive of the theorem. □

3.3.6 DGM Quality analogue to the abc-conjecture

Conjecture 3.17 (abc-conjecture, DGM Analogue)**.** *Let c_0 be a fixed positive integer, and $S_{DGM} = \{q_s(a, b, c_0) : (a, b, c_0)$ is an $abc-$ triple$\}$, and $s_{c_0} = \max(S_{DGM})$. Then,*

$$\limsup_{c\to\infty} s_c = \infty$$

Remark on Conjecture 3.17. *This Conjecture is conditional on the infinitude of Mersenne primes (Conjecture 3.15) or the infinitude of Fermat primes. It is also true if there are infinitely many primes of the form $p^i + q^j$, where p, q are fixed primes. Finally, it is true if there exists some selection of primes $i_1, i_2, i_3, ..., i_m, j_1, j_2, ..., j_n$ such that there are infinitely many primes of the form $\prod_{k=1}^{m} i_k^{a_k} + \prod_{k=1}^{n} j_k^{b_k}$ for some sequences of positive integers $a_1, a_2, ..., a_m, b_1, b_2, ..., b_n$. These statements follow from Theorems 3.8, 3.9, 3.10, 3.12, and 3.14.*

3.4 Average Quality and Running Average Quality

In addition to studying the behavior of the Highest-quality DGM triples, we would like to investigate the *average* rate of growth of the DGM quality. To do so, we define two sequences, the Average Quality and Running Average Quality, that categorize this rate of growth.

Definition 3.18 (Average Quality)**.** *For any fixed positive integer c_0, let D be the set of all the DGM Qualities of abc-triples of the form (a, b, c_0). Then, we define (t_{c_0}), the Average DGM Quality, as the mean of D*

$$t_{c_0} = \frac{\sum_{x\in D} x}{|D|}$$

Thus, t is a sequence of positive real numbers.

Definition 3.19 (Running Average Quality)**.** *Let c_0 be a fixed, positive integer, and let S_{DGM} be the set of all the DGM Qualities of abc-triples (a, b, c) with $c \leq c_0$. Then, r_{c_0}, the Running Average Quality, is the mean of S_{DGM}; that is,*

$$r_{c_0} = \frac{\sum_{x\in S_{DGM}} x}{|S_{DGM}|}$$

Similar to t, r is also a sequence of positive real numbers.

The Average Quality (t) simply returns the mean of the qualities of all possible abc-triples (a, b, c_0) with a fixed value of c_0, so it does not offer a clear picture on the asymptotic average behavior of the quality. However, the Running Average Quality (r) allows us to study the asymptotic behavior of the quality.

Figures 3 and 4 show the Average Quality (t) and the Running Average Quality (r). Figure 3 only shows the sequences t_{c_0}, r_{c_0} for $c_0 \leq 100$, but Figure 4 displays a larger data set as it graphs the sequences for all $c_0 \leq 1000$. The Average Quality is almost as unpredictable as the DGM Quality itself, but appears to have a bit more regularity as most of the values lie in the interval $[0.5, 0.55]$. The Running Average Quality appears to converge to a limit. We were experimentally determine that this limit is approximately 0.528, and we would like to prove this in the future.

We were able to prove the following bound for the Running Average Quality.

Theorem 3.20. *If $\lim_{c_0\to\infty} r_{c_0}$ exists, then it is greater than $\frac{1}{3}$.*

Proof. Assume that $\lim_{c_0\to\infty} r_{c_0}$ exists. Let c_0 be a fixed positive integer. Consider the set S_{DGM}, which is the the set of all the DGM Qualities of abc-triples (a, b, c) with $c \leq c_0$. Then, every element of the set S_{DGM} must be greater than $\frac{1}{3}$, by Theorem 3.4. This means the mean of S_{DGM} is greater than $\frac{1}{3}$, so $r_{c_0} > \frac{1}{3}$. Hence, by the squeeze theorem, we have $\lim_{c_0\to\infty} r_{c_0} > \lim_{c_0\to\infty} \frac{1}{3} = \frac{1}{3}$. □

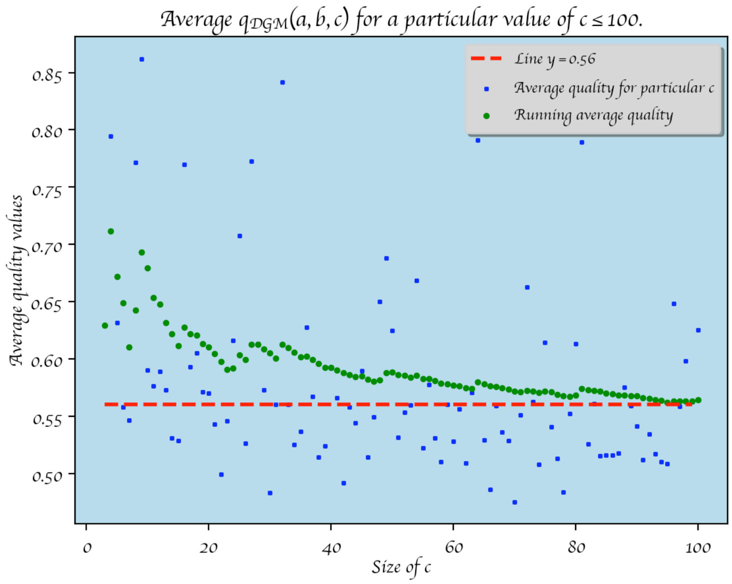


Figure 3: The Average Quality and Running Average Quality evaluated on all abc-triples with $c \leq 100$.

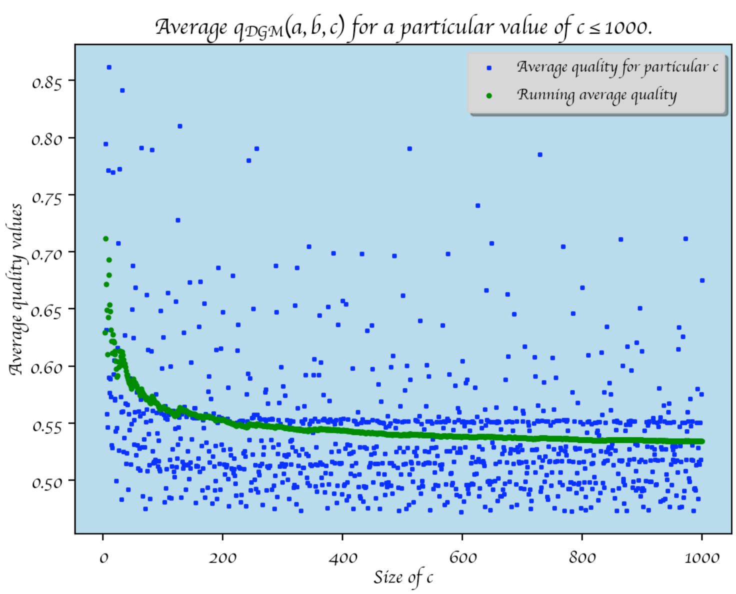


Figure 4: The Average Quality and Running Average Quality evaluated on all abc-triples with $c \leq 1000$.

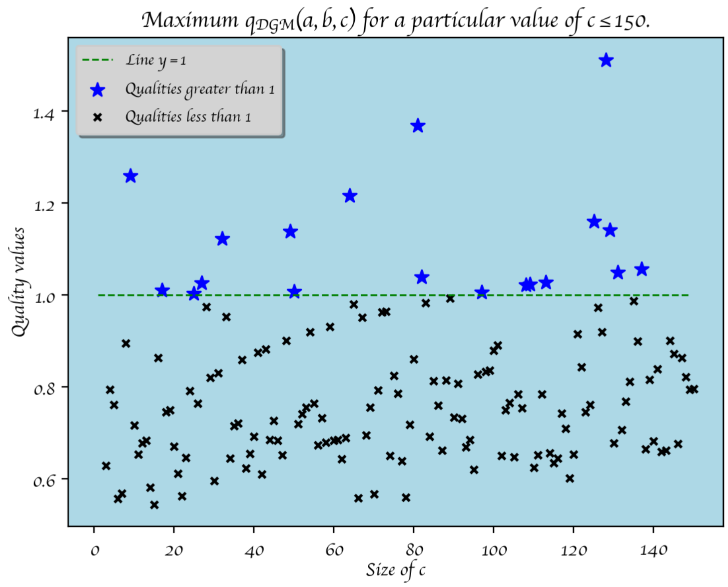


Figure 5: DGM quality evaluated on all abc-triples with $c \leq 150$, only showing triples with highest possible quality. Blue stars correspond to DGM Hits.

Remark. *The Average Quality is also bounded below by $\frac{1}{3}$, also due to Theorem 3.4. However, we would like to improve this Theorem to a better bound in the future.*

3.5 Experimental Analysis of DGM Quality Triples

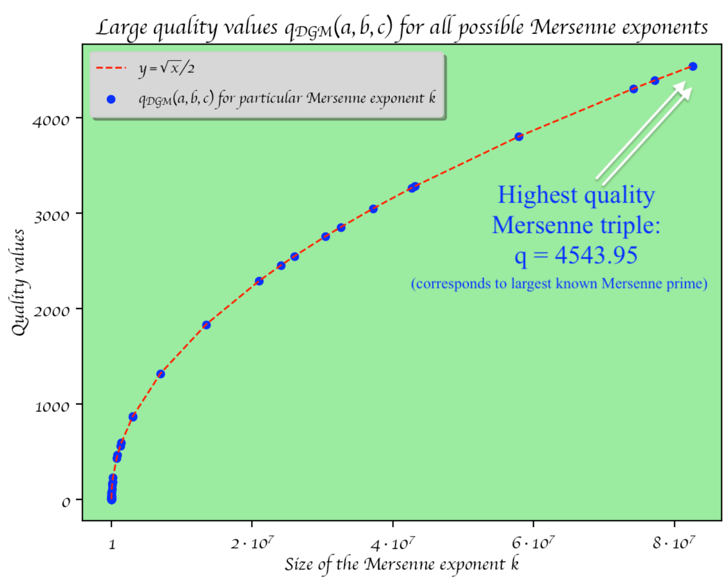
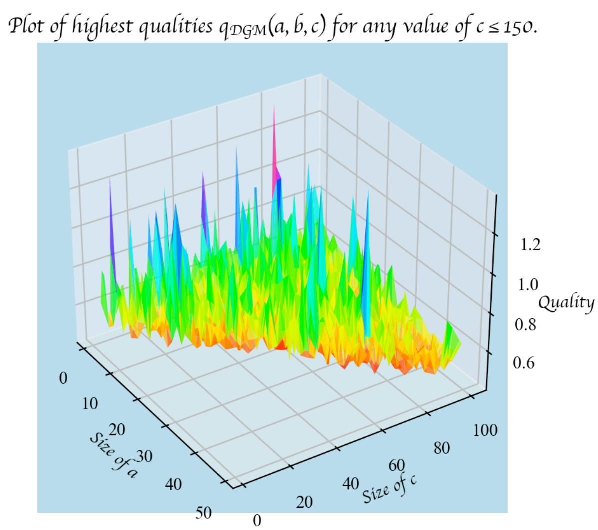


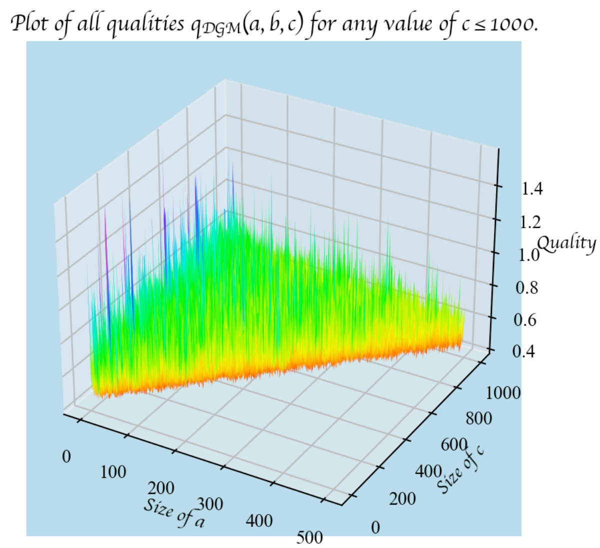
Figure 6: Investigating the DGM quality evaluated on Mersenne primes.

We will begin by presenting several graphs to visualize the DGM quality. Such graphs were created using Python programs and Matplotlib (see Section 6 for algorithm details and more experimental results).

Figure 5 shows 150 triples (a, b, c) for $c \in [1, 150]$. For any fixed c, we analyze all possible triples and only show the triple with highest quality, which is s_c. Notice that this figure contains some regularity, even though this regularity could not be theoretically proven. That is, there is a series of dots that appear to be trending upwards logarithmically. We note that these dots appear to become more sparse as we go out towards larger c, and these triples (a, b, c) that



(a) 3D graph of quality for $c \leq 150$.



(b) 3D graph of quality for $c \leq 1000$.

Figure 7: Graph of quality for all possible abc-triples with c less than a fixed constant.

correspond to these dots almost always satisfy $c \in \mathbb{P}$. The reason that triples (a, b, c) with prime values of c tend to have reasonably high DGM Quality is because c only contributes *one* prime towards ω, so ω is expected to be small, which implies that the triple has reasonably large quality for its size. The points in blue in this figure correspond to DGM Hits. The highest-quality triple picture in Figure 5 is the Mersenne triple $(1, 127, 128) = (1, 2^7 - 1, 2^7)$, with a quality of $q_{DGM}(a, b, c) = \frac{\sqrt{7}}{2} \approx 1.3239$.

Figure 6 demonstrates the quality evaluated on all possible Mersenne triples, and shows the largest possible quality that we obtained, approximately 4543.95. The x-axis in this figure does *not* correspond to the actual Mersenne triple itself, since the sizes of these triples grow quite large. Instead, the x-axis plots the corresponding Mersenne exponent for any Mersenne triple so it really represents a logarithmic scale with respect to the size of the Mersenne triples.

Finally, Figures 7a and 7b show another way to visualize the quality in three dimensions. The x- and y- axes show the values of a and c (b can be calculated directly from the values of a, c) and the z-axis represents the quality. The spikes in this graph correspond to high-quality triples. As c increases, the frequency of high-quality triples tends to decrease, as demonstrated through Figure 7b, but there are more triples with quality greater than 1.4.

We will now include some of the high-quality triples that we calculated using the DGM Quality.

Table 2 includes all triples $(1, b, c)$, where $c \leq 140,000$ and $q_{DGM}(1, b, c) \geq 1.6$. Triples in blue correspond to Mersenne triples, where b is prime and c is a power of 2. The triple in red is a Fermat triple, which is, in fact, the largest known Fermat triple $(1, 65536, 65537)$. The maximum value of ω in this table is $\omega = 4$, which is why we consider all of the triples in this table extremely "round."

Table 3 considers the highest-quality abc-triples using the Standard Quality, which are also shown in Table 1; however, the table shows both the Standard Quality of these triples and the corresponding DGM Quality. It is noteworthy that several of the best abc-triples of the Standard Quality actually have reasonably small DGM Quality for their size. Although all of these triples are DGM Hits, *none* of them are high-quality DGM Triples (as none have quality greater than 5). This is due to some of the high-quality abc-triples, like the 7th-highest-quality triple (which is the worst triple, using the standard quality, on the table) having a large ω value, which significantly reduces their DGM quality. Similarly, several DGM triples, like the Mersenne triples, are not even abc-hits. Thus, there is not a direct correlation between high-quality triples for the standard metric and the DGM metric.

a	b	c	ω	DGM Quality $q_{DGM}(a, b, c)$
1	4374	4375	4	1.68658
1	8191	8192	2	1.80279
1	8192	8193	3	1.65067
1	10368	10369	3	1.60809
1	12288	12289	3	1.62772
1	13121	13122	3	1.63528
1	15551	15552	3	1.65475
1	17496	17497	3	1.66819
1	18432	18433	3	1.67412
1	23327	23328	3	1.70077
1	27647	27648	3	1.71987
1	39366	39367	3	1.75926
1	52488	52489	3	1.79101
1	57121	57122	3	1.70978
1	59048	59049	4	1.6593
1	59049	59050	4	1.60059
1	62207	62208	3	1.80963
1	65536	65537	2	2 (rounded)
1	73727	73728	3	1.82815
1	78124	78125	3	1.68747
1	79999	80000	3	1.61747
1	81919	81920	3	1.61973
1	131071	131072	2	2.06155
1	131072	131073	3	1.95287
1	139967	139968	3	1.8972
1	139968	139969	3	1.8972

Table 2: All triples with quality greater than 1.6 where $a = 1, c \leq 140,000$, using the DGM Quality.

a	b	c	**Standard Quality**	**DGM Quality**	ω
2	$3^{10} \cdot 109$	23^5	1.6299	2.1424	4
11^2	$3^2 \cdot 5^6 \cdot 7^3$	$2^{21} \cdot 23$	1.6260	1.8226	6
$19 \cdot 1307$	$7 \cdot 29^2 \cdot 31^8$	$2^8 \cdot 3^{22} \cdot 5^4$	1.6235	2.0388	8
283	$5^{11} \cdot 13^2$	$2^8 \cdot 3^8 \cdot 17^3$	1.5808	1.9809	6
1	$2 \cdot 3^7$	$5^4 \cdot 7$	1.5679	1.6866	4
7^3	3^{10}	$2^{11} \cdot 29$	1.5471	1.8386	4
$7^2 \cdot 41^2 \cdot 311^3$	$11^{16} \cdot 13^2 \cdot 79$	$2 \cdot 3^3 \cdot 5^{23} \cdot 953$	1.5444	1.9180	10

Table 3: Evaluating all of the best abc-triples for the standard quality using the DGM Quality. Many triples, like the 5th highest-quality triple, have surprisingly low DGM Quality.

4 DGM Quality Class

4.1 Motivation and Definition

The DGM Quality is an effective metric to characterize the roundness of a triple and it is easier to analyze theoretically. However, it is not a perfect metric because the quality metric can not be strictly controlled as it is possible to make the DGM Quality well over 4000. The standard quality, on the other hand, likely does not have any values above 2. Thus, in order to make the DGM Quality behave similarly to the Standard Quality (in an asymptotic sense), we would like to define a new *class* of quality metrics. In particular, we would like to parametrize the DGM Quality using a few dials, and vary these dials in order to define new quality metrics.

First, we can control the growth of the quality by simply raising the denominator of the quality to a fixed parameter, β. A larger β value makes the quality grow slower. Second, we can define a parameter α, which allows for finer control of the quality by tuning the influence of ω. Instead of raising the primes to the power of ω in the DGM Quality, we raised the primes to ω^α, where α is a positive real number. A larger value of α reduces the growth of the quality, while also enhancing the influence of ω. Using these new parameters α, β, we create the following class:

Definition 4.1 (DGM Quality Class)**.** *Let $\alpha \geq 0, \beta > 0$ be real numbers. Let (a, b, c) be an abc-triple. Consider the set P of the primes dividing abc ($P = \{p : p|abc, p \in \mathbb{P}\}$). Let $\omega = |P|$, the number of primes dividing the product abc. We define $U = \{p_i^{\omega^\alpha} : p_i \in P\}$.*

Then, for each value of α, β, we create a quality metric that belongs to the DGM Quality Class as follows:

$$q_C(a, b, c; \alpha, \beta) = \frac{\ln(c)}{\Big(\ln(DGM(U))\Big)^\beta} = \frac{\ln(c)}{\Big(\ln(DGM\{p_1^{\omega^\alpha}, p_2^{\omega^\alpha}, ..., p_\omega^{\omega^\alpha}\})\Big)^\beta}$$

Note: if we specify $\alpha = 1, \beta = 1$, then, we obtain $q_C(a, b, c; 1, 1) = q_{DGM}(a, b, c)$ (the DGM Quality).

4.2 Simple Properties and Analysis

First, we can rewrite the formula for this class in a simpler way, which allows us to compute the quality quickly.

Theorem 4.2. *For every abc-triple, and for every $\alpha \geq 0, \beta > 0$, let ω be the number of primes dividing abc and let (p_i) be the sequence of primes dividing abc. Then,*

$$q_C(a, b, c; \alpha, \beta) = \frac{\ln(c)}{\left(\omega^\alpha \cdot \sqrt[w]{\prod_{i=0}^{\omega} \ln(p_i)}\right)^\beta}$$

Proof. Using Definition 4.1, we know that

$$\begin{aligned} q_C(a, b, c; \alpha, \beta) &= \frac{\ln(c)}{\Big(\ln(DGM\{p_1^{\omega^\alpha}, p_2^{\omega^\alpha}, ..., p_\omega^{\omega^\alpha}\})\Big)^\beta} \\ &= \frac{\ln(c)}{\ln\left(\exp\left(\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega} \ln\ln(p_i^{\omega^\alpha})\right)\right)\right)^\beta} \end{aligned}$$

by the definition of the DGM (Definition 3.1). Thus,

$$\begin{aligned} q_C(a, b, c; \alpha, \beta) &= \frac{\ln(c)}{\left(\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega} \ln\ln(p_i^{\omega^\alpha})\right)\right)^\beta} \\ &= \frac{\ln(c)}{\left(\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega} \ln(\omega^\alpha \cdot \ln(p_i))\right)\right)^\beta} \\ &= \frac{\ln(c)}{\left(\exp\left(\frac{1}{\omega}\left(\sum_{i=1}^{\omega} \ln(\omega^\alpha) + \ln(\ln(p_i))\right)\right)\right)^\beta} \\ &= \frac{\ln(c)}{\left(\exp\left(\frac{1}{\omega}\left(\sum_{i=1}^{\omega} \alpha\ln(\omega) + \ln(\ln(p_i))\right)\right)\right)^\beta} \\ &= \frac{\ln(c)}{\left(\exp\left(\frac{\omega\alpha\cdot\ln(\omega)}{\omega} + \frac{1}{\omega}\sum_{i=1}^{\omega} \ln(\ln(p_i))\right)\right)^\beta} \end{aligned}$$

so

$$
\begin{aligned}
q_C(a,b,c;\alpha,\beta) &= \frac{\ln(c)}{\left(\exp(\alpha\ln(\omega))\cdot\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega}\ln(\ln(p_i))\right)\right)^{\beta}} \\
&= \frac{\ln(c)}{\left(\omega^{\alpha}\cdot\exp\left(\frac{1}{\omega}\sum_{i=1}^{\omega}\ln(\ln(p_i))\right)\right)^{\beta}} \\
&= \frac{\ln(c)}{\left(\omega^{\alpha}\cdot\sqrt[w]{\prod_{i=1}^{\omega}\exp\left(\ln(\ln(p_i))\right)}\right)^{\beta}} \\
&= \frac{\ln(c)}{\left(\omega^{\alpha}\cdot\sqrt[w]{\prod_{i=1}^{\omega}\ln(p_i))}\right)^{\beta}}
\end{aligned}
$$

□

Next, we will use this theorem to prove a lower bound for the quality values that depends on the value of ω.

Theorem 4.3. *For all abc-triples (a,b,c), and for all $\alpha \geq 0, \beta > 0$, let ω be the number of distinct primes dividing abc. Then,*

$$
q_C(a,b,c;\alpha,\beta) > \frac{\ln(c)}{\left(\omega^{\alpha-1}\cdot\ln(abc)\right)^{\beta}}
$$

Proof. Using Theorem 4.2 and the arithmetic-geometric mean inequality, we know that

$$
\begin{aligned}
q_C(a,b,c;\alpha,\beta) &= \frac{\ln(c)}{\left(\omega^{\alpha}\cdot\sqrt[w]{\prod_{i=1}^{\omega}\ln(p_i))}\right)^{\beta}} \\
&\geq \frac{\ln(c)}{\left(\omega^{\alpha}\cdot\frac{1}{\omega}\sum_{i=1}^{\omega}\ln(p_i))\right)^{\beta}} \\
&> \frac{\ln(c)}{\left(\omega^{\alpha-1}\cdot\sum_{i=1}^{\omega}\ln(p_i))\right)^{\beta}} \\
&\geq \frac{\ln(c)}{\left(\omega^{\alpha-1}\cdot\ln(abc)\right)^{\beta}} \text{ by Lemma 3.3}
\end{aligned}
$$

□

This theorem has a corollary that is the DGM Class analogue of Theorem 3.4:

Corollary 4.3.1. *For all abc-triples (a,b,c), and for all $\alpha \in [0,1], \beta \geq 1$, we know that*

$$
q_C(a,b,c;\alpha,\beta) > \frac{1}{3(\omega^{\alpha-1})^{\beta}} > \frac{1}{3}
$$

where ω is the number of primes dividing abc.

Proof. Let a,b,c be an abc-triple. Let $\alpha \in [0,1], \beta \in (0,1]$. By Theorem 4.3,

$$\begin{aligned}
q_C(a,b,c;\alpha,\beta) &> \frac{\ln(c)}{\left(\omega^{\alpha-1}\cdot\ln(abc)\right)^{\beta}} \\
&> \frac{\ln(c)}{\left(\omega^{\alpha-1}\cdot 3\ln(c)\right)^{\beta}} \\
&\geq \frac{\ln(c)}{3\ln(c)\left(\omega^{\alpha-1}\right)^{\beta}} \\
&= \frac{1}{3(\omega^{\alpha-1})^{\beta}} \\
&\geq \frac{1}{3\cdot(1)^{\beta}} \\
&= \frac{1}{3}
\end{aligned}$$

□

We now use these results to investigate how changing α, β influences the behavior of the quality and the resulting high-quality triples.

4.3 Varying alpha

For the purpose of this section, we will fix the value of $\beta = 1$ in order to focus on the α parameter of the DGM Quality Class.

We note that $\alpha = 1$ corresponds to the DGM Quality metric investigated in Section REF DGM. As we increase α past 1, the significance of the ω in the quality of a triple simply increases. Thus, the resulting high-quality triples for these new metrics have very similar prime factorizations to the high-quality DGM triples that we have already considered in Section REF DGM. Instead, we would like to *decrease* α and investigate the behavior of the quality metrics as $\alpha \to 0$.

First, we will prove that increasing α directly reduces the growth of the quality metric.

Theorem 4.4. *Let $\alpha_1, \alpha_2 : 0 \leq \alpha_1 < \alpha_2$. Fix $\beta > 0$. Then, for all abc-triples (a,b,c), $q_C(a,b,c;\alpha_1,\beta) > q_C(a,b,c;\alpha_2,\beta)$.*

Proof. Let (a,b,c) be an abc-triple. Since $\omega \geq 1$, we know that $\omega^{\alpha_1} < \omega^{\alpha_2}$. Thus,

$$\left(\omega^{\alpha_1}\cdot\sqrt[w]{\prod_{i=1}^{\omega}\ln(p_i)}\right)^{\beta} < \left(\omega^{\alpha_2}\cdot\sqrt[w]{\prod_{i=1}^{\omega}\ln(p_i)}\right)^{\beta}$$

By Theorem 4.2, we have

$$\begin{aligned}
q_C(a,b,c;\alpha_1,\beta) &= \frac{\ln(c)}{\left(\omega^{\alpha_1}\cdot\sqrt[w]{\prod_{i=1}^{\omega}\ln(p_i)}\right)^{\beta}} \\
q_C(a,b,c;\alpha_2,\beta) &= \frac{\ln(c)}{\left(\omega^{\alpha_2}\cdot\sqrt[w]{\prod_{i=1}^{\omega}\ln(p_i)}\right)^{\beta}}
\end{aligned}$$

Hence,

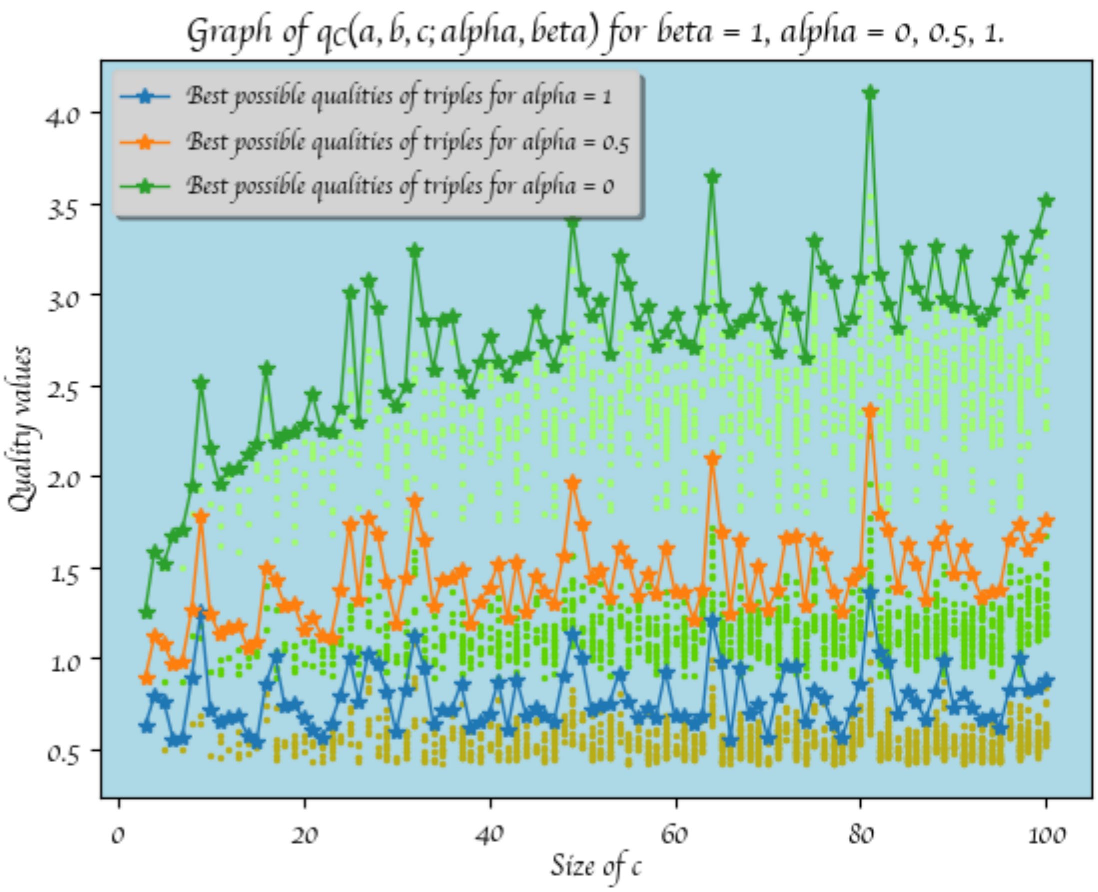


Figure 8: Comparing the behavior of the DGM Quality Class on different values of α, namely $\alpha = 0, 0.5, 1$.

$$q_C(a, b, c; \alpha_1, \beta) > q_C(a, b, c; \alpha_2, \beta)$$

□

Thus, $\alpha = 0$ should allow the quality metric to grow the fastest. We will focus on the interesting metric created when $\alpha = 0, 0.5$: $q_C(a, b, c, 0, 1)$.

We will first investigate $\alpha = 0$. The resulting quality metric can be calculated using the following expression, by Theorem 4.2:

$$q_C(a, b, c; 0, 1) = \frac{\ln(c)}{\sqrt[w]{\prod_{i=1}^{\omega} \ln(p_i))}}$$

This quality metric is the DGM Quality Metric multiplied by ω. We can prove, using Corollary 4.3.1, that this new quality metric is always greater than 1.

Theorem 4.5. *For all abc-triples, the value of $q_C(a, b, c; 0, 1)$ is always greater than 1.*

Proof. There are three cases. The first case is when $\omega = 1$.

When $\omega = 1$, the only possible triple is $(a, b, c) = (1, 1, 2)$, which has quality $\frac{\ln(2)}{\ln(2)} = 1$.

The second case is $\omega = 2$. When $\omega = 2$, the value of one of a or b must be 1. Without loss of generality, assume $a = 1$. Then, we know that $abc = bc \leq c^2$. By Theorem 4.3,

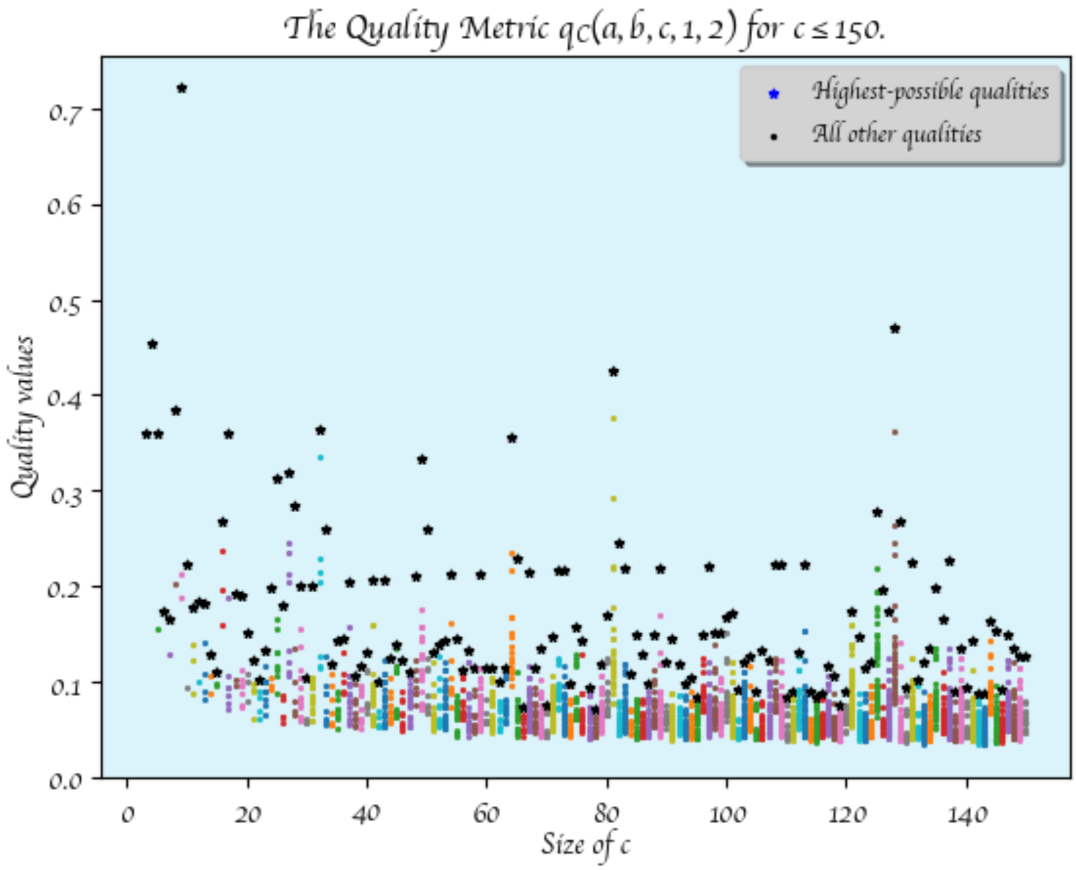


Figure 9: The DGM Class quality metric for $\alpha = 1$, $\beta = 2$. This is DGM Quality, except that the denominator is squared. Triples in black are higher than any other triples with the same x-coordinate

$$
\begin{aligned}
q_{\mathrm{C}}(a,b,c;0,1) &> \frac{\ln(c)}{\left(\omega^{\alpha-1}\cdot\ln(abc)\right)^{\beta}} \\
&> \frac{\ln(c)}{\frac{1}{\omega}\cdot\ln(c^2)} \\
&> \frac{2\omega\ln(c)}{\ln(c))} \\
&= 2\omega \\
&= 4
\end{aligned}
$$

The final case is when $\omega \geq 3$. Then, using Corollary 4.3.1, we have

$$
\begin{aligned}
q_C(a,b,c;0,1) &\geq \frac{1}{3\cdot(\omega)^{(\alpha-1)\cdot\beta}} \\
&= \frac{1}{3\cdot(\omega)^{-1}} \\
&= \frac{\omega}{3} \\
&\geq 1.
\end{aligned}
$$

These three cases account for all possible abc-triples, which proves the theorem. $\square$

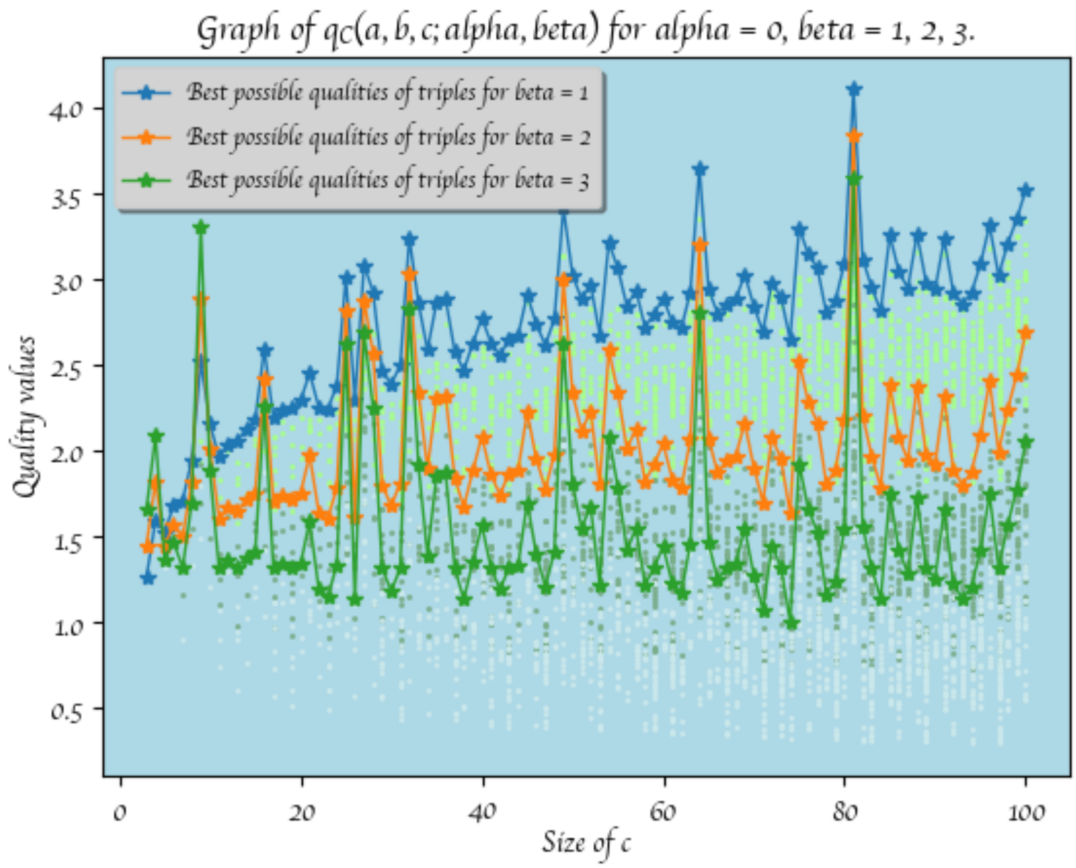


Figure 10: The DGM Class quality metric for $\alpha = 0$, $\beta = 1, 2, 3$. The starred points correspond to triples that are higher than any other triples with the same x-coordinate.

Thus, the $\alpha = 0$ quality is quite different from the DGM Quality and the Standard Quality, because it grows much faster. In particular, we can use Mersenne primes to quantify this.

Theorem 4.6. *For any Mersenne triple* $(a, b, c) = (1, 2^n - 1, 2^n)$, $q_C(a, b, c; 0, 1) \geq \sqrt{n}$.

Proof. Let (a, b, c) be a Mersenne triple with Mersenne exponent n. Then, by Theorem 3.14, $q_{DGM}(a, b, c) = q_C(a, b, c; 1, 1) = \frac{\sqrt{n}}{2}$.

Since $q_C(a, b, c; 0, 1) = \omega \cdot q_C(a, b, c; 1, 1)$ for all abc-triples, and $\omega = 2$ for any Mersenne triple, we have $q_C(a, b, c; 0, 1) = 2 \cdot \frac{\sqrt{n}}{2} = \sqrt{n}$. $\square$

However, this quality is still interesting because it is the *closest* quality metric (in this class) to the Standard Quality. This is precisely because the influence of ω on the quality of a certain triple is reduced when $\alpha = 0$. Another approach that we explored - but do not include here - is to divide the $\alpha = 0$ quality by $\sqrt{\ln(c)}$ or another function with similar growth rate in order to tame the quality's growth.

$\alpha = 0.5$ is also interesting, because it incorporates characteristics of both the DGM High-Quality triples ($\alpha = 1$) with the High-Quality triples for $\alpha = 0$. We present these triples in Section 4.5.

4.4 Varying beta

The significance of β on the quality metrics is less subtle than that of α. Increasing β forces the quality metrics to grow slower. However, β is useful because increasing β (1) restricts the amount of triples that have high-quality, and (2) forces the quality metric to grow quite slowly.

For example, when $\alpha = 1$ (the DGM Quality case), $\beta = 1$ is not sufficient to force the quality to grow slowly - see Theorem 3.14. However, $\beta = 2$ *is* sufficient, and we conjecture that this quality is bounded above but also does not go to 0 extremely quickly. That is, we claim that $\beta = 2$ creates a quality that is as similar to the Standard Quality as possible. We now generalize this conjecture for *all* α.

Conjecture 4.7. *For every $\alpha \geq 0$, there exists some $\beta > 0$ such that the quality metric given by $q_C(a, b, c; \alpha, \beta)$ is bounded above, and*

$$\limsup_{c \to \infty} \max\{q_C(a, b, c; \alpha, \beta) : a, b, c \text{ is an abc-triple}\} = L$$

for some finite, nonzero L.

In the following section, we will use graphical analysis to investigate the behavior of high-quality triples when we change the value of α, β.

4.5 Experimental Evaluation

We first investigate the effect of α on the growth of $q_C(a, b, c; \alpha, 1)$ in Figure 8. This figure demonstrates how a larger α results in the quality metric growing slower. The points with stars in this figure correspond to triples (a, b, c_0) that have highest possibly quality for a fixed value of c_0. Changing α does not change the relative position of these starred points significantly, but it does privilege certain triples more than others. For example, the triple $(1, 80, 81)$ is considered the absolute best when $\alpha = 0$, but it *remains* the absolute best triples when $\alpha = 0.5$ and 1.

Figure 9 shows just one element of this class: $q_C(a, b, c, 1, 2)$. This particular quality metric corresponds to the DGM quality, except that we square the denonimator in order to make the quality grow slower. Notice that, in this figure, most of the quality values are quite low, and there are only very few triples with quality greater than even 0.5. This is why $\beta = 2$ is sufficient to force the DGM Quality to grow slowly. The high-quality triples when $\beta = 2$ are simply a refined subset of the high-quality triples when $\beta = 1$ (the DGM Quality). We will demonstrate this in Table 7.

Finally, Figure 10 shows $q_C(a, b, c; 1, \beta)$ for $\beta = 1, 2, 3$. As β increases, the quality metrics grow slower, and the number of high-quality triples decreases. However, decreasing β also makes the quality metric a lot more unpredictable, as evidenced through the tall spikes corresponding to triples such as $(1, 63, 64)$, $(1, 80, 81)$, and $(1, 48, 49)$.

We now present a few high-quality triples for various values of α, β. Tables 4, 5, and 6 all show similar data. They investigate the metrics $q_C(a, b, c; 0.5, 1)$, $q_C(a, b, c; 1, 1)$, and $q_C(a, b, c; 0, 1)$, respectively. In particular, the tables show triples (a, b, c) where $a = 1, c \leq 200000$ and the quality of $(1, b, c)$ is higher than the quality of any other triple $(1, b_0, c_0)$ where $c_0 \leq c$. Let us call these triples *record-setting consecutive triples*, since b and c will be consecutive integers. There is a significant amount of overlap between Tables 4 and 6, which means that $\alpha = 0.5$ and $\alpha = 0$ are fairly similar, whereas $\alpha = 1$ is slightly different. These tables suggest that $\alpha = 0.5$ is likely the best of these three metrics, as it privileges triples that are *moderately*, but not extremely, round. Finally, Table 7 graphs the record-setting consecutive triples for all $c \leq 200000$, but includes different values of β $(0.75, 1, 1.25, 1.5, 1.75, 2, 3)$. As we increase β, the set of record-setting consecutive triples becomes smaller and smaller. When we go as far as $\beta = 3$, we have reduced the number of consecutive record-setting triples to just a few extremely special ones. Each time we increase β, we obtain a *subset* of the previous set of triples. We would like to prove this result in the future.

5 Incorporating Smoothness

5.1 Motivation and Definition

Despite the DGM Quality class measuring many different characteristics of a triple (a, b, c), there are certain metrics in this class that allow triples to use significantly large primes (the DGM Quality, $q_C(a, b, c; 1, 1)$ is one such metric). Such triples are not considered *smooth*. We would like to explicitly incorporate the smoothness of a triple into the quality. To do so, we divide by the largest prime used in abc, raised to some power ψ. This allows us to create the following Smooth DGM Quality Class.

Definition 5.1 (Smooth DGM Quality Class)**.** *Let $\alpha \geq 0, \beta > 0, \psi \geq 0$ where $\alpha, \beta, \psi \in \mathbb{R}$. Let P be the set of primes dividing abc, and let $w = |P|$. Let $U = \{p_i^{\omega^\alpha} : p_i \in P\}$.*

Let p_{max} be the largest prime divisor of abc (that is, $p_{max} = \max(P)$). Then, for every value of α, β, ψ, we create a quality metric in the Smooth DGM Quality Class as

a	b	c	$q_C(a,b,c;0.5,1)$
1	2	3	0.89623982
1	3	4	1.13092874
1	7	8	1.3468996
1	8	9	1.79247965
1	24	25	1.9613958
1	48	49	2.25966876
1	63	64	2.41473076
1	80	81	2.67772184
1	224	225	2.74781055
1	242	243	3.01721265
1	512	513	3.23868058
1	1215	1216	3.32598641
1	2400	2401	3.94896109
1	4374	4375	4.25337653
1	25920	25921	4.29015347
1	59048	59049	4.45792947
1	71874	71875	4.57921043
1	123200	123201	4.64804663
1	137780	137781	4.72150024
1	156249	156250	4.72767218

Table 4: All triples $(1,b,c)$, where $c \leq 200000$, that have quality higher than any other triple $(1,b_0,c_0)$ with $c_0 \leq c$, for the metric $q_C(a,b,c;0.5,1)$.

a	b	c	$q_C(a,b,c;1,1)$
1	2	3	0.629476469
1	3	4	0.794310867
1	7	8	0.89524643
1	8	9	1.258952938
1	80	81	1.3687847
1	242	243	1.498050638
1	512	513	1.589248129
1	4374	4375	1.686580318
1	8191	8192	1.80278785
1	62207	62208	1.809626069
1	65536	65537	2.000001376
1	131071	131072	2.06155348

Table 5: All triples $(1,b,c)$, where $c \leq 200000$, that have quality higher than any other triple $(1,b_0,c_0)$ with $c_0 \leq c$, for the metric $q_C(a,b,c;1,1)$ (DGM Quality).

$$q_{SC}(a,b,c;\alpha,\beta,\psi) = \frac{\ln(c)}{\Big(\ln(\mathit{DGM}(U))\Big)^{\beta} \cdot p_{max}^{\psi}} = \frac{\ln(c)}{\Big(\ln(\mathit{DGM}\{p_1^{\omega^\alpha}, p_2^{\omega^\alpha}, ..., p_\omega^{\omega^\alpha}\})\Big)^{\beta} \cdot p_{max}^{\psi}} = \frac{q_C(a,b,c;\alpha,\beta)}{p_{max}^{\psi}}$$

When $\psi = 0$ or ψ is very close to 0, the smoothness does not have a large influence on the quality metric. However, for $\psi \geq 0.5$, the smoothness has a large impact on which abc-triples are designated high-quality triples. Any value of ψ greater than 1 does not result in a very interesting quality metric, as the quality values very quickly get small. Increasing ψ makes the resulting quality metric grow slower.

We will analyze $q_{SC}(a,b,c;\alpha,\beta,\psi)$ using computational methods in the following section. In the future, we would like to perform more theoretical analysis on the class q_{SC}.

5.2 Analysis

We first present a plot, in Figure 11, of $q_{SC}(a,b,c;1,1,0.5)$ for small abc-triples ($c \leq 100$). The value of $\psi = 0.5$ in this metric is large enough to pull all of the DGM quality values towards 0. This is why most of the starred

a	b	c	$q_C(a,b,c;0,1)$
1	2	3	1.25895294
1	3	4	1.58862173
1	5	6	1.67429377
1	6	7	1.70683635
1	7	8	1.79049286
1	8	9	2.51790588
1	15	16	2.59082098
1	24	25	3.00785001
1	48	49	3.41367271
1	63	64	3.64792426
1	80	81	4.1063541
1	224	225	4.35832864
1	242	243	4.49415191
1	512	513	4.76774439
1	624	625	4.83479968
1	675	676	4.89370978
1	1215	1216	5.1537676
1	2400	2401	6.26348502
1	4374	4375	6.74632127
1	25920	25921	6.7960907
1	59535	59536	6.9644818
1	71874	71875	7.17216006
1	123200	123201	7.49186405

Table 6: All triples $(1,b,c)$, where $c \leq 200000$, that have quality higher than any other triple $(1,b_0,c_0)$ with $c_0 \leq c$, using the metric $q_C(a,b,c;0,1)$

a	b	c	$\beta=0.75$	$\beta=1$	$\beta=1.25$	$\beta=1.5$	$\beta=1.75$	$\beta=2$	$\beta=3$
1	2	3	1.2168	1.2590	1.3026	1.3477	1.3944	1.4427	1.6533
1	3	4	1.5354	1.5886	1.6437	1.7006	1.7595	1.8205	2.0862
1	4	5	1.5448						
1	5	6	1.7029	1.6743	1.6462				
1	6	7	1.7637	1.7068	1.6518				
1	7	8	1.8587	1.7905	1.7248				
1	8	9	2.4336	2.5179	2.6051	2.6954	2.7888	2.8854	3.3065
1	15	16	2.6351	2.5908					
1	24	25	3.0593	3.0079	2.9573	2.9076	2.8587		
1	44	45	3.1100						
1	48	49	3.5274	3.4137	3.3036	3.1971	3.0940	2.9943	
1	63	64	3.7695	3.6479	3.5303	3.4165	3.3063	3.1997	
1	80	81	4.1766	4.1064	4.0373	3.9695	3.9027	3.8371	3.5856
1	224	225	4.6016	4.3583	4.1279				
1	242	243	4.7254	4.4942	4.2742	4.0650			
1	512	513	5.0996	4.7677	4.4575	4.1674			
1	624	625	5.1936	4.8348	4.5008	4.1899			
1	675	676	5.2569	4.8937	4.5556	4.2409	3.9479		
1	1215	1216	5.5842	5.1538	4.7565	4.3899	4.0516		
1	1700	1701	5.5854						
1	2303	2304	5.5857						
1	2375	2376	5.6237						
1	2400	2401	6.6131	6.2635	5.9323	5.6187	5.3216	5.0402	4.0559
1	4374	4375	7.1229	6.7463	6.3896	6.0518	5.7318	5.4288	4.3685
1	25920	25921	7.5153	6.7961					
1	50624	50625	7.5319						
1	59535	59536	7.8065	6.9645					
1	71874	71875	8.0145	7.1722	6.4184				

Table 7: For different values of β, we show the record-setting triples consecutive $(1,b,c)$ where $c \leq 200000$. Every time we increase β, we obtain a subset of the previous set of consecutive record-setting triples.

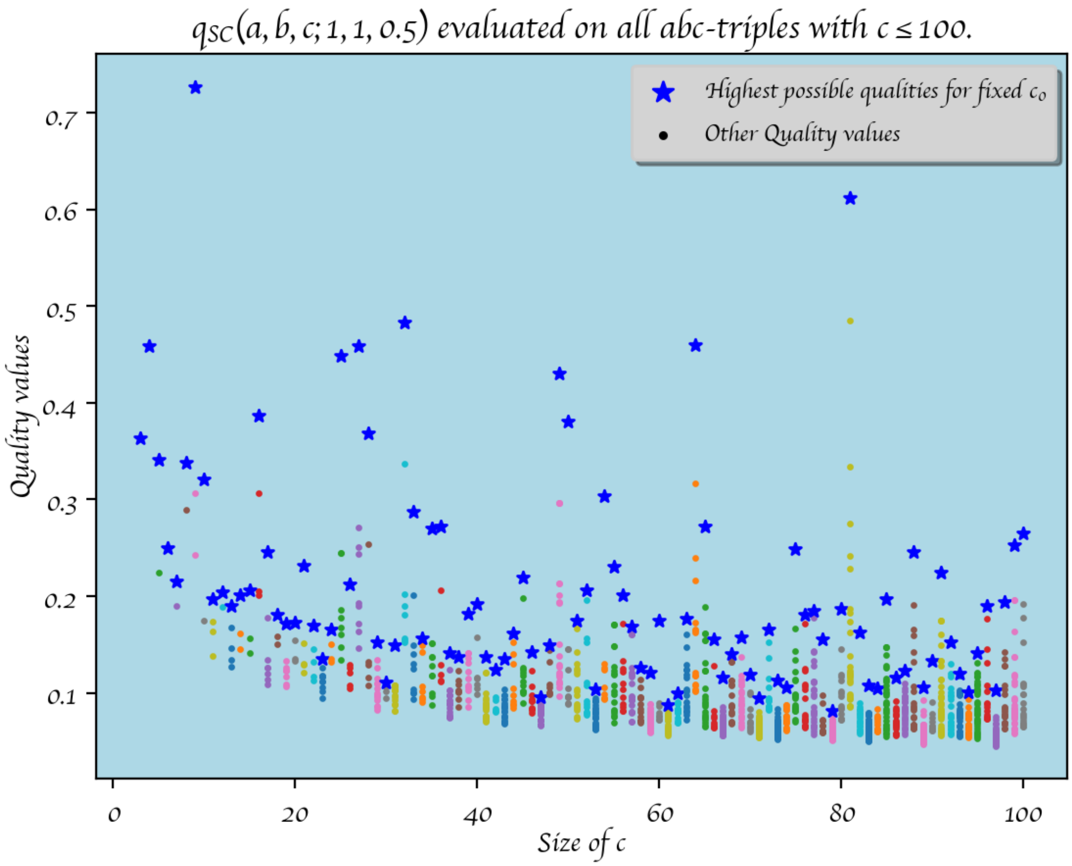


Figure 11: The Smooth DGM Class quality metric for $\alpha = 1$, $\beta = 1$, $\psi = 0.5$, evaluated on all abc-triples with $c \leq 100$. Starred points correspond to triples that are higher than any other triples with the same x-coordinate.

points, which correspond to triples (a, b, c_0) that have maximum possible quality for the size of c_0, lie in the range $q_{SC}(a, b, c; 1, 1, 0.5) \in [0.1, 0.3]$. Nevertheless, there are a few triples with reasonably large quality, such as $(1, 8, 9)$ and $(1, 80, 81)$.

The next plot, Figure 12, shows $q_{SC}(a, b, c; 0.5, 1, 0.25)$. In this plot, we reduce the value of α from Figure 11, which allows the quality to grow slightly faster. We also reduce the value of ψ to 0.25. These changes allow us to create a quality metric that accounts for a triple's roundness, the size of the primes used in the triple when compared to $\ln(c)$, and the smoothness of the triple. We believe this metric is quite similar to the Standard Quality in its asymptotic behavior, as evidenced in Figure 12.

6 Algorithms

In this section, we present three algorithms used to compute high-quality abc-triples using the DGM Quality. Throughout this section, we will not be using any other quality metric besides $q_{DGM}(a, b, c)$. The goal of the algorithms in this section is to find a set of high-quality triples less than a fixed positive integer h.

6.1 Brute Force Algorithm

The simple approach towards finding high-quality triples less than h is to calculate the qualities of all possible abc-triples (a, b, c), with $c \leq h$. This method is listed in Algorithm 1.

This brute force algorithm finds all abc-triples with $c \leq h$ and $q_{DGM}(a, b, c) \geq q_{\min}$ for some fixed positive real number $q_{\min}$. However, since the triples (a_0, b_0, c_0) and (b_0, a_0, c_0) have the same quality, it only checks half of these abc-triples: the triples with $a \leq \lceil \frac{c}{2} \rceil$. For each such triple, it calculates the primes dividing a, b, c, and then uses

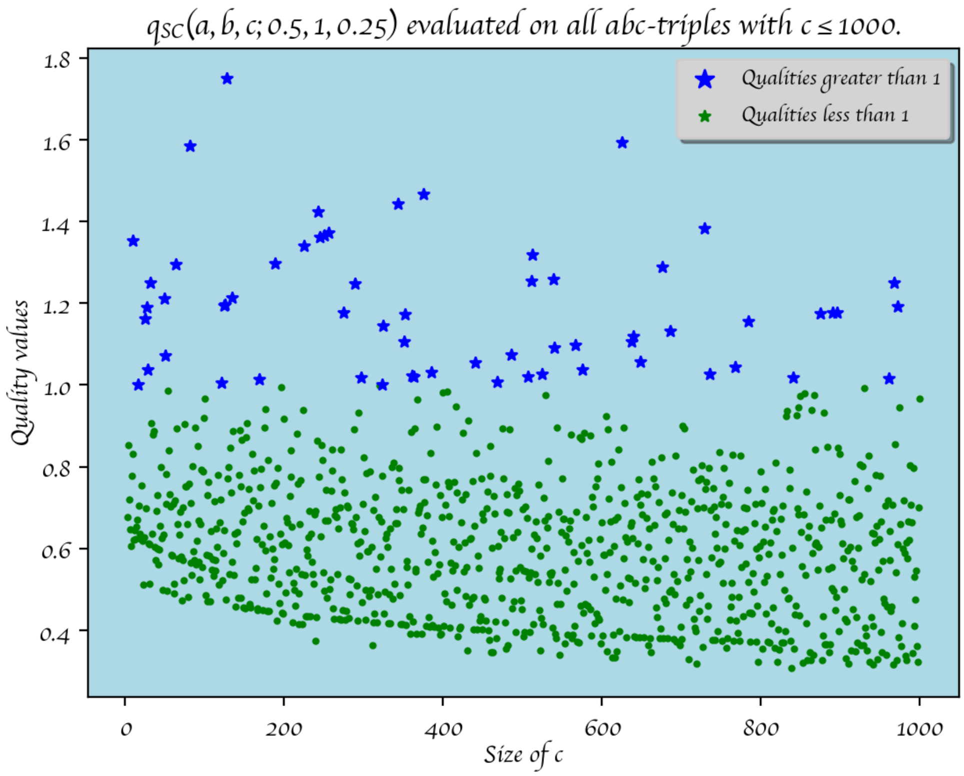


Figure 12: The Smooth DGM Class quality metric for $\alpha = 0.5$, $\beta = 1$, $\psi = 0.25$, evaluated on all abc-triples with $c \leq 100$. The only points shown are the qualities of triples (a, b, c_0) with highest possible quality for a fixed c_0.

Equation 6 to calculate the quality. This, however, requires calculating the prime factorization of a, b, c. Thus, the limiting operations in this algorithm, which consume the most amount of time, are:

1. Determine the prime factorization of abc.
2. Compute the quality for abc-triples (a, b, c) with c less than or equal to h and a less than $\frac{c}{2}$.

In order to find the time complexity of this program, we will have to find the number of abc-triples with $c \leq h$. For any particular value of $c_0 \geq 3$, the number of abc-triples (a, b, c_0) is exactly $\phi(c)$, Euler's Totient Function[1]. However, the algorithm only needs to check half of these triples, so the number of *distinct* quality values for any particular c_0 must be exactly $\frac{\phi(c_0)}{2}$.

Thus, to find the total number of distinct quality values of abc-triples with $c \leq 100$, we must find the sum of $\frac{\phi(c_0)}{2}$ for all positive integers $c_0 \in [3, 100]$. We can do this using the Totient Summatory Function $\Phi(n)$, which defined as follows:

Definition 6.1 (Totient Summatory Function)**.** *For any positive integer n, the Totient Summatory Function $\Phi(n)$ is*

$$\Phi(n) = \sum_{k=1}^{n} \phi(n)$$

Using this function, we can prove the following theorem about the number of abc-triples that Algorithm 1 needs to check.

Theorem 6.2. *For any positive integer h, the number of abc-triples that Algorithm 1 must check is exactly*

[1]Euler's Totient Function $\phi(n)$ counts the number of positive integers $k < n$ that are relatively prime to n.

Algorithm 1 Brute Force Algorithm

Input: h, maximum possible value of c in the abc-triple (a, b, c); q_{min}, minimum threshold for a triple to be designated a high-quality triple

Output: Triple$_q$, Triples$_{a,b,c}$: two lists of numbers that record all triples with quality greater than q_{min} and their corresponding qualities.

```
c ← 3
Triple_q ← []
Triples_{a,b,c} ← []
primes_a ← []
primes_b ← []
primes_c ← []
while c ≤ h do
    ▷ Find all primes dividing c and store in list.
    primes_c ← primefact(c)
    a ← 1
    while a ≤ ⌊c/2⌋ do
        if gcd(a, c) = 1 then
            b ← c − a
            primes_a ← primefact(a)
            primes_b ← primefact(b)
            ▷ Concatenate the list of primes
            primes_{a,b,c} ← primes_a + primes_b + primes_c
            Calculate q_DGM(a, b, c) using Equation 6 and primes_{a,b,c}
            if q_DGM(a, b, c) ≥ q_min then
                Triples_{a,b,c} ← Triples_{a,b,c} + [a, b, c]
                Triple_q ← Triples_q + q_DGM(a, b, c)
            end if
        end if
        a ← a + 1
    end while
    c ← c + 1
end while
```

$$\frac{\Phi(h) - 2}{2}$$

Proof. Let h be a positive integer. We wish to find the number of abc-triples (a, b, c) with $c \leq h$ and $a \leq \lfloor \frac{c}{2} \rfloor$. For any fixed c with $3 \leq c \leq h$, the number of abc-triples (a, b, c) with $a \leq \lfloor \frac{c}{2} \rfloor$ is exactly $\frac{\phi(c_0)}{2}$, by the symmetry of the Totient function. Thus, the total number of abc-triples that Algorithm 1 must consider is the sum of $\frac{\phi(c_0)}{2}$ as c_0 goes from 3 to h:

$$\begin{aligned}
\sum_{k=3}^{h} \frac{\phi(k)}{2} &= \frac{1}{2} \sum_{k=3}^{h} \phi(k) \\
&= \frac{1}{2}(\Phi(h) - \Phi(2)) \\
&= \frac{1}{2}(\Phi(h) - 2) \\
&= \frac{\Phi(h) - 2}{2}
\end{aligned}$$

□

However, the Totient Summatory Function grows on the order of $n^2 + n\ln(n)$ [25]:

$$\Phi(n) \sim \frac{1}{2\zeta(2)} \cdot n^2 + O(n\ln(n)) = \frac{3}{\pi^2} \cdot n^2 + O(n\ln(n)) > \frac{3}{\pi^2} \cdot n^2 > 0.3n^2$$

where $\zeta(s)$ is the Riemann Zeta Function and O represents big-O notation. Thus, the number of triples that we must check is at least

$$\frac{\Phi(h) - 2}{2} > 0.15h^2$$

Thus, the number of abc-triples that the algorithm must find is proportional to h^2. In addition, we still need to compute the prime factorization of a, b, *and* c for all of these triples, which requires at least $0.45h^2$ prime factorization calculations. Even if we used the most efficient known prime factorization algorithm, the General Number Field Sieve, then each of the prime factorization calculations for a triple (a, b, c) would require at least

$$\exp\left(\left(\sqrt[3]{\frac{64}{9}} + o(1)\right)(\ln(abc))^{\frac{1}{3}}(\ln\ln(abc))^{\frac{2}{3}}\right) > \exp\left((\ln(abc))^{\frac{1}{3}}(\ln\ln(abc))^{\frac{2}{3}}\right)$$

calculations [26]. However, qw have to perform this calculation for *every single one* of the (at least) $0.15h^2$ abc-triples. Thus, this algorithm is an exponential time algorithm in the number of digits of h and has complexity at least $O(10^d)$, where d is the number of digits of h.

Due to the large number of abc-triples that need to be considered, finding high-quality triples using this algorithm is quite challenging. This algorithm, implemented in Python, was only able to find triples with very few digits. When h is 1000, our program takes an average of 16.41 seconds to terminate, and the highest-quality triple found was $(93, 2094, 2187)$, with DGM Quality 1.6489. This algorithm takes an unreasonable amount of time to run for $h = 10000$ or more. We would like to develop a better approach than the brute force algorithm, which we do in the following section.

6.2 Power of p, q Method

In Theorem 3.9, we showed that a certain construction of triples works quite well when constructing high-quality DGM Triples. We pick two primes p, q and let a, b be integer powers of these primes. Then, if c is prime or has very few prime factors, the corresponding triple is very likely to have high quality. This observation is the basis for the Power of p, q Algorithm, Algorithm 2.

This algorithm finds all abc-triples, with $c \leq h$ that can be constructed using the Power of p, q method. For each such triple, it calculates the quality. This quality calculation still requires *one* prime factorization calculation for each triple, but this is a large improvement from Algorithm 1, which requires *three* calculations.

Furthermore, this algorithm only requires considering a very small subset of triples. The total number of abc-triples in this algorithm is exactly $i_{\text{max}} \cdot j_{\text{max}}$, where i_{max} and i_{max} are the largest possible powers of p, q in any triple with $c \leq h$. However, we know by construction that $i_{\text{max}} \leq \log_p(\frac{h}{2})$ and $j_{\text{max}} \leq \log_q(\frac{h}{2})$. This is because if $i_{\text{max}} > \log_p(\frac{h}{2})$ *and* $j_{\text{max}} > \log_q(\frac{h}{2})$, then the algorithm checks some triple (a, b, c) where $c = a+b = p^{i_{\text{max}}} + q^{j_{\text{max}}} > p^{\log_p(\frac{h}{2})} + q^{\log_q(\frac{h}{2})} = \frac{h}{2} + \frac{h}{2} = h$, which is not permitted.

Thus, the total number of abc-triples that we check is exactly

$$\log_p\left(\frac{h}{2}\right) \cdot \log_q\left(\frac{h}{2}\right) = \frac{\log_{10}(h) - \log_{10}(2)}{\log_{10}(p)} \cdot \frac{\log_{10}(h) - \log_{10}(2)}{\log_{10}(q)} < C \cdot \log_{10}(h)^2$$

where $C = \frac{1}{\log_{10}(p) \cdot \log_{10}(q)}$. So, if d is the number of digits of h, then this part of the algorithm is an $O(d^2)$ algorithm, which is a *polynomial* time algorithm - a large improvement on the Brute Force version. However, the entire algorithm itself is still not a polynomial algorithm in h, since we must compute a prime factorization for each of these d^2 triples.

Algorithm 2 Power of p, q algorithm

Input: h, maximum possible value of c in the abc-triple (a, b, c); q_{min}, minimum threshold for a triple to be designated a sufficiently high-quality triple; p, q, two distinct prime numbers.

Output: Triple$_q$, Triples$_{a,b,c}$,: two lists of numbers that record all triples with quality greater than q_{min} and their corresponding qualities.

$i_{\text{max}} \leftarrow \lceil \log_p(\frac{h}{2}) \rceil$
$j_{\text{max}} \leftarrow \lceil \log_q(\frac{h}{2}) \rceil$
$i \leftarrow 1$
$j \leftarrow 1$
Triple$_q \leftarrow []$
Triples$_{a,b,c} \leftarrow []$
primes$_a \leftarrow [p]$
primes$_b \leftarrow [q]$
primes$_c \leftarrow []$
while $i \leq i_{\text{max}}$ **do**
    while $j \leq j_{\text{max}}$ **do**
        $a \leftarrow p^i$
        $b \leftarrow q^j$
        $c \leftarrow a + b$
        primes$_c \leftarrow$ primefact(c)
        primes$_{a,b,c} \leftarrow$ primes$_a$ + primes$_b$ + primes$_c$
        Calculate $q_{DGM}(a, b, c)$ using Equation 6 and primes$_{a,b,c}$
        if $q_{DGM}(a, b, c) \geq q_{\text{min}}$ **then**
            Triples$_{a,b,c} \leftarrow$ Triples$_{a,b,c} + [a, b, c]$
            Triple$_q \leftarrow$ Triples$_q + q_{DGM}(a, b, c)$
        end if
    end while
end while

There is no known polynomial-time prime factorization algorithm. Even using the General Number Field Sieve could not make Algorithm 2 a polynomial-time algorithm. Nevertheless, this algorithm is still more efficient than Algorithm 1.

In particular, the algorithm is finished in 0.77 seconds when $h = 10^9$ (median of 5 trials), which allows us to find four triples with DGM quality greater than 10. We will improve on this value in the following section.

6.3 Mersenne Prime Algorithm

Algorithm 3, which is the most specialized and efficient algorithm of the three evaluated, calculates high-quality DGM hits using *only* Mersenne triples. The algorithm is given a known list of the 51 Mersenne primes, and it simply has to calculate the qualities of all of the 51 Mersenne triples (a, b, c) with $c \leq h$ satisfying $q_{DGM}(a, b, c) \geq q_{\text{min}}$. However, there is no prime factorization calculation involved since the prime factorizations of a, b, c are already known. This algorithm must perform at most $\log_2(h)$ steps, which - if d is the number of digits of h - is a *linear* time algorithm in d ($O(d)$). It is much more efficient and always takes less than a second.

The algorithm allows us to find triples with as many as 24 million digits (which correspond to the largest known Mersenne triples), and quality approximately 4543.95, in less than a second (median runtime for any $h \leq 2^{82,589,933}$ is 0.00168).

This algorithm is the most efficient one, but it focuses on a very small selection of triples. We would like to use more sophisticated techniques, such as continued fractions and elliptic curves, to develop algorithms for other quality metrics, including the Smooth DGM Quality Class, which do not have rather straightforward algorithms to calculate high-quality triples.

6.4 Experimental Evaluation

We will use an Apple MacBook Air, with a 1.1 GHz Quad-Core Intel Core i5 processor, and Python 3.8.2 for our calculations. Table 8 compares the three algorithms in this section and demonstrates that the Mersenne prime algorithm

Algorithm 3 Mersenne Prime Algorithm

Input: h, maximum possible value of c in the abc-triple (a, b, c); q_{min}, minimum threshold for a triple to be designated a sufficiently high-quality triple.

Output: Triple$_q$, Triples$_{a,b,c}$,: two lists of numbers that record all triples with quality greater than q_{min} and their corresponding qualities.

$n_{\text{max}} \leftarrow \log_2(h)$
MersenneList $\leftarrow$ List of Mersenne Primes (precalculated)
Mersenne$_{\text{max}}$ $\leftarrow$ MersenneList.index(n_{max})
Triple$_q$ $\leftarrow$ []
Triples$_{a,b,c}$ $\leftarrow$ []
$i \leftarrow 0$
while $i \leq$ Mersenne$_{\text{max}}$ **do**
  $b \leftarrow$ MersenneList[i]
  $c \leftarrow b + 1$
  $q_{DGM}(a, b, c) \leftarrow \frac{\sqrt{\log_2(c)}}{2}$
  if $q_{DGM}(a, b, c) \geq q_{\text{min}}$ **then**
    Triples$_{a,b,c}$ $\leftarrow$ Triples$_{a,b,c}$ + $[a, b, c]$
    Triple$_q$ $\leftarrow$ Triples$_q$ + $q_{DGM}(a, b, c)$
  end if
  $i \leftarrow i + 1$
end while

is the most efficient in terms of computation time, the number of factorizations that need to be performed, the largest quality found, and the largest triple. The most inefficient aspect of Algorithms 1 and 2 is that a large number of prime factorization calculations are needed. Despite Algorithm 2 having a quadratic complexity in terms of the number of prime factorizations that are needed, this is still enough to force the algorithm to be exponential. In fact, any algorithm that requires just d prime factorizations, where d is the number of digits of h, is exponential in d - since there is no known classical, polynomial-time prime factorization algorithm.

It is somewhat unfair to compare the three algorithms, as the Mersenne Prime method allows us to find high-quality triples while avoiding the prime factorization problem, through Theorem 3.14. The results here demonstrate the power of this theorem. However, even when h is just $10,000$, the Brute Force algorithm has to perform $10^{2b} = 10^{10}$ prime factorization calculations. Although these prime factorizations are all performed on numbers with less than five digits, these calculations still take a significant amount of time. So if we let $h = 2^{82,589,933}$, as in the Mersenne case, the computational requirements would be infeasible. The second algorithm allows us to reduce the number of prime factorizations down to $b^2 = 81$ calculations when $h = 10^9$, but such calculations are being performed on much larger triples. This is why the second algorithm would also fail if we let $h = 2^{82,589,933}$, since we would have to perform over $(24,000,000)^2$ prime factorization calculations on numbers with thousands, or millions, of digits. Thus, the Mersenne algorithm is a huge shortcut, but only works for the DGM Quality metric that we defined.

7 Future Extensions to Other Quality Measures

We also defined two other quality metrics that were quite different in their behavior from the DGM Quality, DGM Quality Class, and Smooth DGM Quality Class. We will conduct more analysis on these metrics in the future, but present some of our initial investigations here.

7.1 Divisor Quality

One quality metric that we defined is called the Divisor Quality (DQ). This metric privileges triples that have extremely few divisors. The metric uses the Divisor Function $d(n)$, which calculates the number of positive integer divisors of n.

Definition 7.1. *The Divisor Quality (DQ) q_D equals $\frac{\ln(c)}{\ln(d(abc))}$ for all abc-triples (a, b, c).*

It is fairly easy to find high-quality triples using the divisor quality. If a, b, c are all prime (which requires $a = 2$ or $b = 2$), then $d(abc)$ is always $d(a) \cdot d(b) \cdot d(c) = 2 \cdot 2 \cdot 2 = 8$ by the multiplicativity of $d(n)$. Note that such triples are formed when b, c are *twin primes*, which is why we will call these triples Twin Prime Triples. Twin Prime Triples

Property	**Algorithm 1 (Brute Force)**	**Algorithm 2 (Power of p, q)**	**Algorithm 3 (Mersenne)**
Highest DGM Quality	1.6849	10.0424	4543.9502
Number of digits of c in largest triple	5	9	24,000,000
Largest computed c	$10,000$	924291401	$2^{82,589,933}$
Median time taken (5 trials)	16.41 sec.	0.77345 sec.	0.00168 sec
Number of factorizations required	$C \cdot 10^{2d}$	$C \cdot d^2$	0

Table 8: Comparison of the complexity and performance of Algorithms 1, 2, 3 to find large, high-quality triples. Algorithms were implemented using Python 3.8.2 on an Apple Macbook Air. d is the number of digits of h.

are the "best possible" triples for the divisor quality, as we show in the following theorem. They are the equivalent of High-Quality DQ triples.

Theorem 7.2. *For any Twin Prime Triples $(2, b, c)$, where $b, c \in \mathbb{P}$, the divisor quality of $(2, b, c)$ is greater than the divisor quality of any other triple (a_1, b_1, c_1) with $a_1, b_1 \geq 2, c_1 < c$.*

Proof. Let (a, b, c) be a twin prime triple - $a = 2$ and $b, c \in \mathbb{P}$. Let (a_1, b_1, c_1) be a triple with $c_1 < c, a_1, b_1 \geq 2$. Then, $d(a_1b_1c_1) = d(a_1)d(b_1)d(c_1)$ by the multiplicativity of d.

Then $d(a_1)d(b_1)d(c_1) \geq 2^3 = 8$. However, $d(abc) = d(a)d(b)d(c) = 8$. Since $\ln(c) > \ln(c_1)$, we have

$$q_D(a, b, c) = \frac{\ln(c)}{d(a)d(b)d(c)} = \frac{\ln(c)}{8} > \frac{\ln(c_1)}{8} = \frac{\ln(c_1)}{d(a_1)d(b_1)d(c_1)} = q_D(a_1, b_1, c_1)$$

□

We also know that if there are infinitely many twin primes, which is the Twin Prime Conjecture, then the Divisor Quality goes to ∞.

Theorem 7.3. *If the Twin Prime Conjecture holds,*

$$\limsup_{c \to \infty} \max\{q_D(a, b, c) : a, b, c \text{ is an abc-triple}\} = \infty$$

Proof. Assume the Twin Prime Conjecture. Then, there are infinitely many Twin Prime Triples. Since all such triples have quality $\frac{\ln(c)}{8}$, and we can make c arbitrarily large, it follows that we can make the divisor quality larger than any fixed positive integer N_0. The theorem immediately follows. □

7.2 Harmonic Mean Quality

We also briefly investigated a quality created using the Harmonic Mean. The motivation for this quality is that, instead of taking the DGM of the primes dividing abc raised to the power of ω, we take the Harmonic Mean:

$$HM(x_1, x_2, x_3, ..., x_k) = \frac{k}{\sum_{i=1}^{k} \frac{1}{x_k}}$$

Definition 7.4. *The Harmonic Mean quality q_H of any abc-triple is*

$$q_H(a, b, c) = \frac{\ln(c)}{\ln(HM\{p_1^\omega, p_2^\omega, ..., p_\omega^\omega\})}$$

where (p_i) is the sequence of primes dividing abc and ω is the number of primes dividing abc.

The behavior of this quality metric is quite subtle and we would like to conduct more detailed analysis on this quality in the future. The metric privileges triples that use very few primes in their factorizations, like the DGM Quality, but also has some parallels to the Standard Quality, q_s.

8 Conclusion

We were able to define several new quality metrics that judged the roundness of certain abc-triples, while incorporating their prime factorizations and their smoothness. We analyzed these metrics in detail and investigated their asymptotic behavior, and were able to use our analysis to develop algorithms that quickly calculate large, high-quality triples. These algorithms found triples with millions of digits efficiently. We also connected our results to several unsolved problems in mathematics, such as Fermat's Last Theorem, the Lenstra-Pomerance-Wagstaff Conjecture, and the Twin Prime Conjecture. These results provide new insight towards the abc-conjecture and its relation to the behavior of primes and round triples. Finally, we formulated several of our own conjectures about these quality metrics, such as Conjecture 3.17, which are unsolved.

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